1 Continuous functions on topological spaces

Definition 1.1. (Continuous functions of topological spaces.) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a map $f : X \to Y$ is called *continuous* if ...

2 Bases for topological spaces

Definition 2.1. (Basis of a topology.) Let $(X, \mathcal{T}_X)$ be a topological space. Then we say that $\mathcal{B} \subseteq \mathcal{T}$ is a *basis* for the topology $\mathcal{T}$ if every element of $\mathcal{T}$ can be expressed as a union of elements of $\mathcal{B}$. We say that the basis $\mathcal{B}$ *generates* the topology $\mathcal{T}$.

Remark 2.2. By convention, we say that the empty set $\varnothing$ is the union of an empty collection of open sets.

In-class Exercises

1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and fix $y_0 \in Y$. Show that the constant map

$$f : X \to Y$$
$$f(x) = y_0$$

   is continuous.

2. (a) Let X be a set, and let $\mathcal{T}_{\because}$ denote the discrete topology on X. Let $(Y, \mathcal{T}_Y)$ be a topological space. Show that any map $f : X \to Y$ of these topological spaces is continuous.

   (b) Let $(X, \mathcal{T}_X)$ be a topological space. Let Y be a set, and let $\mathcal{T}_{\bullet}$ denote the indiscrete topology on Y. Show that any map $f : X \to Y$ of these topological spaces is continuous.

3. Let (X, d) be a metric space. Show that the set of open balls

$$\mathcal{B} = \{\ B_r(x_0) \mid x_0 \in X, r \in \mathbb{R}, r > 0\ \}$$

   is a basis for the topology induced by d.

4. Let (X, d_X) and (Y, d_Y) be metric spaces. Recall that $X \times Y$ is then a metric space with metric

$$d_{X\times Y} : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

   (a) Show that the set

$$\mathcal{B} = \{ \, U \times V \mid U \subseteq X \text{ is open, } V \subseteq Y \text{ is open } \}$$

   forms a basis for the topology induced by $d_{X\times Y}$. *Hint: Homework # 6 Problem 2(b).*

   (b) Is every open set in $X \times Y$ of the form $U \times V$ with $U \subseteq X$ is open and $V \subseteq Y$ is open ?

5. An advantage of identifying a basis for a topology is that many topological statements can be reduced to statements about the basis. As an example, prove the following theorem.

   Theorem (Equivalent definition of continuity.) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}_Y$ be a basis for $\mathcal{T}_Y$. Prove that a map $f : X \to Y$ is continuous if and only if for every open set $U \in \mathcal{B}_Y$, the preimage $f^{-1}(U) \subseteq X$ is open.

6. **(Optional).**

   (a) Consider the topology on $\mathbb{R}^n$ induced by the Euclidean metric. Prove that the following set is a basis for $\mathbb{R}^n$.

$$\mathcal{B} = \{ \, B_\epsilon(x) \mid \; \epsilon > 0, \, \epsilon \text{ is } \textbf{rational}; \, x \in \mathbb{R}^n, \text{ all coordinates } x_i \text{ of } x \text{ are } \textbf{rational.} \}$$

   (b) Show that $\mathbb{R}^n$ has uncountably many open sets, but that the basis $\mathcal{B}$ is countable.