1 Hausdorff topological spaces; Sequences in topological spaces

Recall from Homework #7, Question 2:

Definition 1.1. (Hausdorff space.) A topological space $(X, \mathcal{T})$ is called *Hausdorff* if for any pair of distinct points x and y, there exist **disjoint** open sets $U \subseteq X$ and $V \subseteq X$ so with $x \in U$ and $y \in V$.

Definition 1.2. (Convergence topological spaces.) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in a topological space $(X, \mathcal{T})$. Then we say that that $(a_n)_{n\in\mathbb{N}}$ converges to a point $a_\infty \in X$, and we write $\lim_{n\to\infty} a_n = a_\infty$, if ...

In-class Exercises

1. (a) Give an example of a topological space $(X, \mathcal{T})$ and a point $x \in X$ such that the set $\{x\}$ is not closed.
   (b) Let $(X, \mathcal{T})$ be a **Hausdorff** topological space, and let $x \in X$. Prove that $\{x\}$ is a closed set.
2. (a) Give an example of a topological space $(X, \mathcal{T})$, and a sequence $(a_n)_{n\in\mathbb{N}}$ in X that converges to (at least) two distinct points $a_\infty \in X$ and $\widetilde{a}_\infty \in X$.
   (b) Now suppose that $(X, \mathcal{T})$ is a **Hausdorff** topological space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that, if $(a_n)_{n\in\mathbb{N}}$ converges, then it converges to only one point a_∞.
3. Let C be a closed subset of a topological space $(X, \mathcal{T})$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in C that converge to a point $a_\infty \in X$. Prove that $a_\infty \in C$.
4. (a) Show that the continuous image of a Hausdorff topological space need not be Hausdorff. Specifically, find topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ such that X is Hausdorff but Y is not Hausdorff, and a continuous **surjective** map $f : X \to Y$.
   (b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and assume that $(Y, \mathcal{T}_Y)$ is Hausdorff. Show that, if there exists a continuous **injective** map $f : X \to Y$, then X must also be Hausdorff.