Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an intuitive geometric explanation of each of the 3 properties that define a metric.
2. Let $X = \{a, b, c\}$. Which of the following functions define a metric on X?

   (a)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 3 \end{aligned}$$

   (b)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 4 \end{aligned}$$

3. Consider the set $\mathbb{Z}$ with the Euclidean metric (defined by viewing $\mathbb{Z}$ as a subset of the metric space $\mathbb{R}$). What is the ball $B_3(1)$ as a subset of $\mathbb{Z}$? What is the ball $B_{\frac{1}{2}}(1)$?
4. Let (X, d) be a metric space, $r > 0$, and $x \in X$. Show that $x \in B_r(x)$. Conclude in particular that open balls are always non-empty.
5. Let (X, d) be a metric space, and suppose that $r, R \in \mathbb{R}$ satisfy $0 < r \leq R$. Show the containment of the subsets $B_r(x) \subseteq B_R(x)$ of X for any point $x \in X$.
6. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$.
   (a) Let x and $r > 0$ be real numbers. Show that $B_r(x)$ is an open interval in $\mathbb{R}$. What are its endpoints?
   (b) Show that every interval of the real line the form (a, b), $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$ is open, for any $a < b \in \mathbb{R}$.
   (c) Show that the interval $[0, 1] \subseteq \mathbb{R}$ is closed.
7. Let (X, d) be a metric space, and let $U \subseteq X$ be a subset. Does the set U necessarily need to be either open or closed? Can it be neither? Can it be both?

Assignment questions

(Hand these questions in!)

1. Let X be a set. A function $f : X \to \mathbb{R}$ is called *bounded* if there is some number $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in X$. Let $\mathcal{B}(X, \mathbb{R})$ denote the set of bounded functions from X to $\mathbb{R}$.
   (a) Show that the function
   $$\begin{aligned} d_\infty : \mathcal{B}(X, \mathbb{R}) \times \mathcal{B}(X, \mathbb{R}) &\longrightarrow \mathbb{R} \\ d_\infty(f, g) &= \sup_{x \in X} |f(x) - g(x)| \end{aligned}$$
   is well-defined, that is, the suprema always exist.
   (b) Show that the function d_∞ defines a metric on $\mathcal{B}(X, \mathbb{R})$.

(c) Explain why the following metric on $\mathbb{R}^n$ is a special case of this construction.

$$d_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$d(\overline{x}, \overline{y}) = \max_{1 \le i \le n} |x_i - y_i|$$

where $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$.

2. Let (X, d) be a metric space. Show that a nonempty subset $U \subseteq X$ is open if and only if U can be written as a union of open balls in X.

3. Let (X, d) be a metric space. Fix $x_0 \in X$ and $r > 0$ in $\mathbb{R}$. Show that the set $\{x \mid d(x_0, x) \le r\}$ is closed.

4. (a) Prove *DeMorgan's Laws*: Let X be a set and let $\{A_i\}_{i \in I}$ be a collection of subsets of X.

$$(i) \quad X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \qquad\qquad (ii) \quad X \setminus \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (X \setminus A_i)$$

Hint: Remember that a good way to prove two sets B and C are equal is to prove that $B \subseteq C$ and that $C \subseteq B$!

(b) Let (X, d) be a metric space, and let $\{C_i\}_{i \in I}$ be a collection of closed sets in X. Note that I need not be finite, or countable! Prove that $\bigcap_{i \in I} C_i$ is a closed subset of X.

(c) Now let (X, d) be a metric space, and let $\{C_i\}_{i \in I}$ be a **finite** collection ($I = \{1, 2, \ldots, n\}$) of closed sets in X. Prove that $\bigcup_{i \in I} C_i$ is a closed subset of X.

5. Let (X, d) be a metric space, and consider $Y \subseteq X$ as a metric space under the restriction of the metric to Y. Show by example that a subset $U \subseteq Y$ that is open in Y may or may not be open in X. For each example you should clearly define the sets X, Y, U, and the metric being used, but you may state without proof whether the set U is open in Y and whether it is open in X.