Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $X = \{a, b, c, d\}$ with the topology
$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$
   (a) Is X connected?
   (b) Is X path–connected?
   (c) Find a proper subset of X that is connected, and a proper subset of X that is disconnected.

2. Let $(X, \mathcal{T})$ be a topological space. Show that any subset $A = \{x\} \subseteq X$ of a single element is connected.

3. (a) Show that, for $a, b \in \mathbb{R}$, the subsets $\varnothing, \{a\}, (a, b), (a, b], [a, b), [a, b], (a, \infty), [a, \infty), (\infty, b), (\infty, b]$, and $\mathbb{R}$ of $\mathbb{R}$ are all intervals in the sense of Problem b.
   (b) Show that every interval must have one of these forms.

4. Let $X = \{0, 1\}$.
   (a) Consider X as a topological space with the discrete topology. Rigorously show that X is not path-connected, and that X is not connected.
   (b) Consider X as a topological space with the indiscrete topology. Rigorously show that X is connected and path-connected.

5. Let $(X, \mathcal{T})$ be a topological space. Suppose that there is a path γ from a point $x \in X$ to a point $y \in X$. Use γ to write down the formula for a path from $y \in X$ to $x \in X$. *Hint:* How can you modify γ in order to traverse the path in the opposite direction?

6. Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Show that there is a path from x to x.

7. Show that $\mathbb{R}$ is path connected: given real numbers a and b, there is a "straight-line" path from a to b given by the function
$$\begin{aligned} \gamma :&[0, 1] \to \mathbb{R} \\ \gamma(t) &= bt + a(t - 1) \end{aligned}$$
   Given points (a_1, a_2) and (b_1, b_2) in $\mathbb{R}^2$, write down the formula for a "straight-line" connecting these points. What about two points in $\mathbb{R}^n$?

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$ be a connected subset. Let B be any subset such that $A \subseteq B \subseteq \overline{A}$. Prove that B is connected.
   Remark: This shows in particular that if A is connected, then so is $\overline{A}$.

2. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a continuous map. Prove that if X is connected, then $f(X)$ is connected. In other words, the continuous image of a connected space is connected.

(b) Recall the Intermediate Value Theorem from real analysis (which you may use without proof).

> **Intermediate Value Theorem.** If $f : [a, b] \to \mathbb{R}$ is continuous and d lies between $f(a)$ and $f(b)$ (i.e. either $f(a) \le d \le f(b)$ or $f(b) \le d \le f(a)$), then there exists $c \in [a, b]$ such that $f(c) = d$.

Define a subset $A \subseteq \mathbb{R}$ to be an *interval* if whenever $x, y \in A$ and z lies between x and y, then $z \in A$.
Prove that any interval of $\mathbb{R}$ is connected. *Hint:* Worksheet #14 Problem 4.

(c) Prove that any subset of $\mathbb{R}$ that is not an interval is disconnected.

These last two results together prove:

> **Theorem (Connected subsets of $\mathbb{R}$).** A subset of $\mathbb{R}$ is a connected if and only if it is an interval.

3. (a) Let $(X, \mathcal{T}_X)$ be a topological space, and $x, y, z \in X$. Suppose that there is a path in X from x to y, and a path from y to z. Show that there is a path from x to z.
Hint: See Homework #7 Problem 2.

(b) Suppose that $\{A_i\}_{i \in I}$ is a collection of path-connected subsets of a topological space $(X, \mathcal{T})$. Show that, if the intersection $\bigcap_{i \in I} A_i$ is nonempty, then the union $\bigcup_{i \in I} A_i$ is path-connected.

4. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are path-connected topological spaces. Show that the product $X \times Y$ with the product topology $\mathcal{T}_{X \times Y}$ is path-connected.

5. (a) Prove the following result.

> **Theorem (Generalized Intermediate Value Theorem).** Let $(X, \mathcal{T}_X)$ be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where the topology on $\mathbb{R}$ is induced by the Euclidean metric). If $x, y \in X$ and c lies between $f(x)$ and $f(y)$, then there exists $z \in X$ such that $f(z) = c$.

(b) Prove that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. (In other words, show that there is some $x \in [0, 1]$ so that $f(x) = x$).
Hint: Consider the function

$$\begin{aligned} g : [0,1] &\to \mathbb{R} \\ g(x) &= f(x) - x. \end{aligned}$$