Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space.
   (a) Let $(X, \mathcal{T})$ be a topological space. Explain why the condition that X is compact is stronger than the assumption that X has a finite open cover.
   (b) Show that every topological space has a finite open cover.
   Hint: What is the first axiom of a topology?

2. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Prove that the two following definitions of compactness are equivalent.
   - The subset A is *compact* if it is a compact topological space with respect to the subspace topology $\mathcal{T}_A$.
   - The subset A is *compact* if it satisfies the following property: for any collection of open subsets $\{U_i\}_{i \in I}$ of X such that $A \subseteq \bigcup_{i \in I} U_i$, there is a finite subscollection $U_1, U_2, \ldots, U_n$ such that $A \subseteq \bigcup_{i=1}^{n} U_i$.

3. Give an example of a subsets $A \subseteq B$ of $\mathbb{R}$ such that ...
   (a) A is compact, and B is noncompact
   (b) B is compact, and A is noncompact

4. Determine the connected components of $\mathbb{R}$ with the following topologies (see Problem 1).
   (a) the topology induced by the Euclidean metric
   (b) the discrete topology
   (c) the indiscrete topology
   (d) the cofinite topology

Assignment questions

(Hand these questions in!)

0. **(Optional).** Submit your Math 490 course evaluation!

1. **Definition (Connected components of a topological space).** Let $(X, \mathcal{T}_X)$ be a topological space. A subset $C \subseteq X$ is called a *connected component* of X if
   (i) C is connected;
   (ii) if C is contained in a connected subset A, then $C = A$.

   (a) Show that any connected component of X is closed. (*Hint*: Homework #10, Problem 1).
   (b) Let $x \in X$. Show that the set
   $$\bigcup_{\substack{A \text{ is a connected set,} \\ x \in A}} A$$
   is a connected component of X.

(c) Show that X is the **disjoint union** of its connected components. In other words, show that every point of X is contained in one, and only one, connected component.

(d) Determine the connected components of $\mathbb{Q}$ (with the Euclidean metric). (Remember to rigorously justify your answer!)

(e) Deduce from the example of $\mathbb{Q}$ that connected components need not be open.

(f) Suppose that X has the property that every point has a connected neighbourhood. Show that the connected components of X are open.

2. Suppose that $(X, \mathcal{T})$ is a topological space, and that C and D are compact subsets.

   (a) Show that $C \cup D$ is compact.

   (b) Suppose that X is Hausdorff. Show that $C \cap D$ is compact.

3. Prove the following result. This theorem is a major reason we care about compactness!

   Theorem (Generalized Extreme Value Theorem). Let X be a nonempty compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has the standard topology). Then $\sup(f(X)) < \infty$, and there exists some $z \in X$ such that $f(z) = \sup(f(X))$. That is, f achieves its supremum on X.

4. (a) Let (X, d) be a metric space. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that contains no convergent subsequence. Prove that, for every $x \in X$, there is some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many points of the sequence.

   (b) Prove that any compact metric space is sequentially compact.

   Combined with Homework #5 Problem 5, this exercise proves:

   Theorem (Compactness vs sequential compactness in metric spaces). Let (X, d) be a metric space. Then X is compact if and only if X is sequentially compact.

   (Neither direction of this theorem holds, however, for arbitrary topological spaces!)

   Combined with Worksheet #8, Problem 2, this exercise proves:

   Theorem (Compactness in $\mathbb{R}^n$). Endow $\mathbb{R}^n$ with the Euclidean metric. A subspace $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

5. **Definition (Regular topological spaces).** A topological space X is called *regular* if, for every point $x \in X$ and every nonempty closed set C that does not contain x, there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.

   (a) Let X be a topological space with the T_1 property. Explain why the condition that X is regular is stronger than the condition that X is Hausdorff. (A space that is both regular and T_1 is said to satisfy the T_3 *property.*)

   (b) Suppose that the topology on a space X is induced by a metric d. Prove that X is regular.

   (c) Suppose that X is a compact, Hausdorff topological space. Prove that X is regular.