Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Give an example of a metric space and a subset that is both open and closed. Give an example of a subset that is neither open nor closed.
   (b) Recite the Topologist Scout Oath:

   "On my honour, I will do my best
   to never claim to prove a set is closed by showing that it is not open,
   and to never claim to prove a set is open by showing that it is not closed."

2. Let X and Y be sets, and $f : X \to Y$ any function. Show that $f^{-1}(Y) = X$, and $f^{-1}(\varnothing) = \varnothing$.

3. Rigorously prove that the following functions are continuous.

   (a) $f(x) = 5$ (b) $f(x) = 2x + 3$ (c) $f(x) = x^2$

   (d) $f(x) = g(x) + h(x)$, for continuous functions g and h.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 + 2$. Find the inverse images of the following sets, and verify that they are open.

   (a) $\mathbb{R}$ (b) $(-1, 1)$ (c) $(2, 3)$ (d) $(6, \infty)$

5. Let (X, d) be a metric spaces. Let $g : X \to X$ be the *identity function*, given by $g(x) = x$ for all $x \in X$. Prove that g is continuous.

6. See the definition of accumulation points and isolated points in Problem (5) below. Let $X = \mathbb{R}$. Find the set of accumulation points and the set of isolated points for each of the following subsets of X.

   (a) $S = \{0\}$ (b) $S = (0, 1)$ (c) $S = \mathbb{Q}$ (d) $S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $A \subseteq X$ and $C \subseteq Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $A \quad \square \quad f^{-1}(f(A)))$ (b) $C \quad \square \quad f(f^{-1}(C)))$

2. **Definition (The discrete metric.)** Given a set X, the *discrete metric on* X is the metric $d_X : X \times X \to \mathbb{R}$ defined by

$$d_X(x, x') = \begin{cases} 0, & x = x' \\ 1, & x \neq x' \end{cases} \qquad \text{for all } x, x' \in X.$$

   Let (X, d) be a metric space with the discrete metric.

(a) Show that, for each $x \in X$, the singleton set $\{x\}$ is open.

(b) Show that **every** subset of X is both open and closed.

(c) Let (Y, d_Y) be any metric space. Prove that **every** function $f : X \to Y$ is continuous.

3. Prove the following theorem.

   Theorem (Equivalent definition of continuity.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if it satisfies the following property: for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed.

4. **Definition (Open map).** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called *open* if for every open set $U \subseteq X$, its image $f(U) \subseteq Y$ is open.

   (a) Give an example of metric spaces (X, d_X) and (Y, d_Y) and a function $f : X \to Y$ that is open, but not continuous.

   (b) Give an example of metric spaces (X, d_X) and (Y, d_Y) and a function $f : X \to Y$ that is continuous, but not open.

   For this question, you may simply state the examples without justification.

5. Consider the following definition.

   Definition (Accumulation points and isolated points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every ball $B_r(x)$ around x contains at least one point of S distinct from x. (Note that x may or may not be an element of S). An element $s \in S$ that is **not** an accumulation point of S is called an *isolated point* of S.

   Let (X, d) be a metric space and let $S \subseteq X$ be a **closed** subset. Let x be an accumulation point of S. Show that x is contained in S.