Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be any functions. What is the relationship between

$$\sup_{x\in\mathbb{R}} f(x) + \sup_{x\in\mathbb{R}} g(x) \qquad \text{and} \qquad \sup_{x\in\mathbb{R}} \Big(f(x) + g(x) \Big) \ ?$$

   Show by example that these values need not be equal.

2. Rigorously determine the limits of the following sequences of real numbers, or prove that they do not converge.

   (a) $a_n = 0$ (b) $a_n = \dfrac{1}{n^2}$ (c) $a_n = n$ (d) $a_n = (-1)^n$

3. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers that converge to a_∞ and b_∞, respectively. Prove that the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $(a_\infty + b_\infty)$.

4. Consider $\mathbb{R}$ with the Euclidean metric. Which of the following maps $f : \mathbb{R} \to \mathbb{R}$ are homeomorphisms (Question 2)?

   (a) $f(x) = ax + b$ (b) $f(x) = x^2$ (c) $f(x) = x^3$ (d) $f(x) = \sin(x)$

5. Consider the sequence $\left(\dfrac{(-1)^n}{n} \right)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Let $\epsilon > 0$ be fixed. Find a number $N \in \mathbb{R}$ so that, for all $m, n \geq N$,

$$\left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| < \epsilon.$$

   This shows that the sequence $\left(\frac{(-1)^n}{n} \right)_{n\in\mathbb{N}}$ is *Cauchy* (as defined in Question 5).

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $C, D \subseteq Y$. For each of the following, determine whether you can replace the symbol $\Box$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f^{-1}(C \cup D) \quad \Box \quad f^{-1}(C) \cup f^{-1}(D)$ (b) $f^{-1}(C \cap D) \quad \Box \quad f^{-1}(C) \cap f^{-1}(D)$

   (c) For $C \subseteq D$, $\quad f^{-1}(D \setminus C) \quad \Box \quad f^{-1}(D) \setminus f^{-1}(C)$

2. Consider the following definition.

   Definition (Homeomorphism.) Let X and Y be metric spaces. Then a map $f : X \to Y$ is a *homeomorphism* if

   - f is continuous;

- f has an inverse f^{-1};
- f^{-1} is continuous.

The metric space X is called *homeomorphic* to Y if there exists a homeomorphism $f : X \to Y$.

(a) Show that, if $f : X \to Y$ is a homeomorphism, then $f^{-1} : Y \to X$ is a homeomorphism. Conclude that X is homeomorphic to Y if and only if Y is homeomorphic to X. (We simply call the two spaces *homeomorphic*.)

(b) Give an example of a function $f : X \to Y$ that is continuous and invertible, but whose inverse f^{-1} is not continuous. You may simply state your example without proof.

Remark: Note the contrast to other mathematical fields, such as linear algebra: if a linear map has an inverse, then the inverse is automatically linear. This exercise shows that this is not true for continuous maps!

3. In this question, we will prove the following result.

Theorem (Another characterization of closed subsets). Let (X, d) be a metric space, and let $A \subseteq X$. Then A is closed if and only if it satisfies the following condition: If $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence of points in A converging to a point $a_\infty \in X$, then the limit a_∞ is contained in A.

(a) Suppose that $A \subseteq X$ is closed. Let a_∞ be the limit of a convergent sequence $(a_n)_{n \in \mathbb{N}}$ of points in A. Show that $a_\infty \in A$.

(b) Suppose that $A \subseteq X$ is a subset that contains the limits of every one of its convergent sequences. Prove that A is closed.

4. Prove the following result:

Theorem (Another definition of continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if the following condition holds: given any convergent sequence $(a_n)_{n \in \mathbb{N}}$ in X, then $\big(f(a_n)\big)_{n \in \mathbb{N}}$ converges in Y, and

$$\lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right).$$

5. Consider the following definition.

Definition (Cauchy sequence.) Let (X, d) be a metric space. Then a sequence $(a_n)_{n \in \mathbb{N}}$ of points in X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists some $N \in \mathbb{R}$ such that $d(a_n, a_m) < \epsilon$ whenever $n, m \geq N$.

(a) Prove that every convergent sequence in X is a Cauchy sequence.

(b) Give an example of a metric space (X, d) and a sequence $(a_n)_{n \in \mathbb{N}}$ in X that is Cauchy but does not converge. Fully justify your solution!