Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Is $\varnothing$ a bounded set?
   (b) Show that any **finite** subset of a metric space is bounded.

2. Give examples of subsets of $\mathbb{R}$ (with the Euclidean metric) that satsify the following.

   (a) open, and bounded
   (b) closed, and bounded
   (c) open, and unbounded
   (d) closed, and unbounded

3. Consider $\mathbb{R}$ with the Euclidean metric. For each of the following sets A, find $\text{Int}(A)$, $\overline{A}$, ∂A, $\text{Int}(\mathbb{R} \setminus A)$, $\overline{\mathbb{R} \setminus A}$, and $\partial(\mathbb{R} \setminus A)$. See Question 3 for the definition of a boundary ∂.

   (a) $\mathbb{R}$ (b) $[0,1]$ (c) $(0,1)$ (d) $\{0,1\}$ (e) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

4. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Find examples of subsets A of $\mathbb{R}$ with the following properties.
   (a) $\partial(A) = \varnothing$
   (b) A has a nonempty boundary, and A contains its boundary ∂A.
   (c) A has a nonempty boundary, and A contains no points in its boundary
   (d) A has a nonempty boundary, and A contains some but not all of the points in its boundary.
   (e) A has a nonempty boundary, and $A = \partial A$.
   (f) A is a **proper** subset of ∂A.

5. Let X be a nonempty set with the discrete metric. Let $A \subseteq X$. Show that $A = \text{Int}(A) = \overline{A}$. Conclude that $\partial A = \varnothing$.

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $A, B \subseteq X$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f(A \cap B) \quad \square \quad f(A) \cap f(B)$
   (b) $f(A \cup B) \quad \square \quad f(A) \cup f(B)$
   (c) For $A \subseteq B$, $\quad f(B \setminus A) \quad \square \quad f(B) \setminus f(A)$

2. (a) Let (X,d) be a metric space, and $A \subseteq X$ a subset. Show that A is bounded if and only if, for every $x \in A$, there is some $R_x > 0$ such that $A \subseteq B_{R_x}(x)$.
   (b) Let (X,d) be a metric space. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in X converging to an element a_∞. Show that the set $\{ a_n \mid n \in \mathbb{N} \}$ is a bounded subset of X.

3. Consider the following definition.

   Definition (Boundary of a set A.) Let (X,d) be a metric space, and let $A \subseteq X$. Then the *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \operatorname{Int}(A)$.

   Let (X,d) be a metric space, and let $A \subseteq X$.

   (a) Prove that $\operatorname{Int}(A) = \overline{A} \setminus \partial A$.

   (b) Prove that $\partial A = \overline{A} \cap \left(\overline{X \setminus A}\right)$.

   (c) Conclude from part (b) that ∂A is closed.

   (d) Additionally conclude from part (b) that $\partial A = \partial(X \setminus A)$.

   (e) Prove the following characteriziation of points in the boundary:

   Theorem (An equivalent definition of ∂A). Let (X,d) be a metric space, and let $A \subseteq X$. Then $x \in \partial A$ if and only if every ball $B_r(x)$ about x contains at least one point of A, and at least one point of $X \setminus A$.

   (f) Deduce that we can classify every point of X in one of three mutually exclusive categories:

   (i) interior points of A;

   (ii) interior points of $X \setminus A$;

   (iii) points in the (common) boundary of A and $X \setminus A$.

4. Prove the following equivalent definition of continuity.

   Theorem (An equivalent definition of continuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f : X \to Y$ is continuous if and only if

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \qquad \text{for every subset } A \subseteq X.$$