Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let (X,d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Recall that we proved that, if $\lim_{n\to\infty} a_n = a_\infty$, then any subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a_∞.
   (a) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence that does not converge. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.
   (b) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence converging to $a \in X$, and a different subsequence converging to $b \in X$, with $a \neq b$. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.
2. Let (X,d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Suppose that the set $\{a_n \mid n \in \mathbb{N}\}$ is unbounded. Explain why $(a_n)_{n\in\mathbb{N}}$ cannot converge.
3. Find examples of sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers with the following properties.
   (a) $\{a_n \mid n \in \mathbb{N}\}$ is unbounded, but $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence
   (b) $(a_n)_{n\in\mathbb{N}}$ has no convergent subsequences
   (c) $(a_n)_{n\in\mathbb{N}}$ is not an increasing sequence, but it has an increasing subsequence
   (d) $(a_n)_{n\in\mathbb{N}}$ has four subsequences that each converge to a distinct limit point
4. Determine which of the following subsets of of $\mathbb{R}^2$ can be expressed as the Cartesian product of two subsets of $\mathbb{R}$.

   (a) $\{(x,y) \mid x \in \mathbb{Q}\}$ (b) $\{(x,y) \mid x > y\}$ (c) $\{(x,y) \mid 0 < y \leq 1\}$ (d) $\{(x,y) \mid x^2 + y^2 < 1\}$
5. Let X be a set with the discrete metric, and consider the product metric on $X \times X$. Show that every subset is both open and closed.

Assignment questions

(Hand these questions in!)

1. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Determine the interior, closure, and boundary of the subset $\mathbb{Q} \subseteq \mathbb{R}$. Remember to rigorously justify your solution!
2. For sets X and Y, let $A, B \subseteq X$ and $C, D \subseteq Y$. Consider the Cartesian product $X \times Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.
   (a) $(A \times C) \cup (B \times D) \quad \square \quad (A \cup B) \times (C \cup D)$
   (b) $(A \times C) \cap (B \times D) \quad \square \quad (A \cap B) \times (C \cap D)$
   (c) $(X \setminus A) \times (Y \setminus C) \quad \square \quad (X \times Y) \setminus (A \times C)$
3. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $C \subseteq X$ and $D \subseteq Y$ are closed subsets. Prove or find a counterexample: the subset $C \times D \subseteq X \times Y$ is closed with respect to the product metric.

4. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be a metric spaces, and suppose that $f : Z \to X$ and $g : Z \to Y$ are continuous functions. Prove that the function

$$(f \times g) : Z \longrightarrow X \times Y$$
$$(f \times g)(z) = \Big(f(z), g(z)\Big)$$

is continuous.

5. (a) **Definition (Open cover).** A collection $\{U_i\}_{i \in I}$ of open subsets of a metric space X is an *open cover* of X if $X = \bigcup_{i \in I} U_i$. In other words, every point in X lies in some set U_i.

Definition (Lebesgue number of an open cover). Let (X, d) be a metric space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Then $\delta > 0$ is a *Lebesgue number*[1] for $\mathcal{U}$ if, for every $x \in X$, there is some associated index $i_x \in I$ such that $B_\delta(x) \subseteq U_{i_x}$.

Suppose that (X, d) is a sequentially compact metric space. Prove that any open cover of X has a Lebesgue number $\delta > 0$.

(b) **Definition (ϵ-nets of a metric space).** Let (X, d) be a metric space. A subset $A \subseteq X$ is called an ϵ-net if $\{B_\epsilon(a) \mid a \in A\}$ is an open cover of X.

Suppose that (X, d) is a sequentially compact metric space, and $\epsilon > 0$. Prove that X has a finite ϵ-net.

(c) Let (X, d) be a sequentially compact metric space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Show that there exists some finite collection $U_{i_1}, \ldots, U_{i_n} \in \mathcal{U}$ so that $\{U_{i_1}, \ldots, U_{i_n}\}$ covers X, i.e., so that $X = U_{i_1} \cup \cdots \cup U_{i_n}$.

We will return to these results later in the course when we study *compactness*.

[1] Named for Henri Lebesgue, `https://en.wikipedia.org/wiki/Henri_Lebesgue`.