Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an example of a metric space (X, d) and a continuous function $f : X \to \mathbb{R}$ such that f has a finite supremum on X, but f does not achieve its supremum at any point $x \in X$.

2. Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set. Show that any subset of S is bounded.

3. Let (X, d) be a metric space. Verify that the collection $\mathcal{T}_d$ of open sets in this metric space does indeed form a topology on the set X.

4. Find all possible topologies on the set $X = \{0, 1\}$.

5. Let $X = \{0, 1, 2\}$. Show that the collection of subsets $\Big\{\varnothing, X, \{0, 1\}, \{1, 2\}\Big\}$ is **not** a topology on X.

6. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show by induction that the intersection of any **finite** collection of open sets is open.
   (b) Explain why this argument does not apply to an infinite collection of open sets.

7. Let X be a set. Show that the discrete topology on X is induced by the discrete metric on X.

8. Let X be a set. See the definition of the cofinite topology on X in Assignment Problem 3. Show that if X is a finite set, then the cofinite topology coincides with the discrete topology on X.

9. Let X be a topological space consisting of either 0 or 1 points. Explain why X satisfies both the T_1 and T_2 properties.

Assignment questions

(Hand these questions in!)

1. Prove the following theorem. This theorem is one of the important reasons we care about sequential compactness!

   Theorem (Extreme value theorem for metric spaces). Let (X, d) be a metric space and C a sequentially compact subset of X. Let $f : X \to \mathbb{R}$ be a continuous function. Then there is a point $c \in C$ so that

   $$f(c) = \sup_{x \in C} f(x).$$

   In other words, prove that f achieves its supremum on C.

2. Let (X, d_X) and (Y, d_Y) be metric spaces.
   (a) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of points in X and Y, respectively. Show that the sequence of points $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ in $(X \times Y, d_{X\times Y})$ converges if and only if both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge.
   Hint: Worksheet # 7 Problems 2(b) and 3(a).

(b) Show that $X \times Y$ is sequentially compact if and only if both X and Y are.

3. Let X be a set, and let $\mathcal{T}$ be the collection of subsets

$$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X \mid X \setminus U \text{ is a finite set}\}.$$

Verify that $\mathcal{T}$ is a topology on X. It is called the *cofinite topology.*

4. **Definition (Subspace topology).** Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be any subset. Then S inherits the structure of a topological space, defined by the topology

$$\mathcal{T}_S = \{U \cap S \mid U \in \mathcal{T}_X\}.$$

The topology $\mathcal{T}_S$ on S is called the *subspace topology.*

Verify that $\mathcal{T}_S$ is in fact a topology.

5. (a) **Definition (T_1-space).** Let $(X, \mathcal{T})$ be a topological space. Then we say that $(X, \mathcal{T})$ has the T_1 *property*, or call X a T_1*-space*, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists some neighbourhood U_x of x that does not contain y, and there exists some neighbourhood U_y of y that does not contain x.

Prove that $(X, \mathcal{T})$ is a T_1-space if and only if, for every point $x \in X$, the singleton set $\{x\}$ is closed. (Mathematicians often refer to this property by the slogan "points are closed").

(b) **Definition (Hausdorff property; T_2-space).** Let $(X, \mathcal{T})$ be a topological space. Then we say that $(X, \mathcal{T})$ has the T_2 *property*, or call X a *Hausdorff* [1] *space*, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists neighbourhoods U_x of x and U_y of y such that U_x and U_y are disjoint.

Show that every space satisfying the T_2 property must satisfy the T_1 property. Conclude in particular that, in a Hausdorff space, singleton sets $\{x\}$ are closed.

(c) Conside the set $\mathbb{R}$ with the cofinite topology (Problem 3). Verify that ($\mathbb{R}$, cofinite) is a T_1-space but not Hausdorff. This shows that the T_2 condition is strictly stronger than the T_1 condition.

(d) Conclude that the cofinite topology is not metrizable. *Hint:* Worksheet #4 Problem 2(a).

[1] named for Felix Hausdorff, `https://en.wikipedia.org/wiki/Felix_Hausdorff`