Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space. Let $S \subseteq X$ and let $\mathcal{T}_S$ be the subspace topology on S. Prove that if S is an open subset of X, and if $U \in \mathcal{T}_S$, then $U \in \mathcal{T}$.

2. (a) Suppose that $(X, \mathcal{T})$ is a topological space with the property that the singleton set $\{x\}$ is open for ever $x \in X$. Prove that $\mathcal{T}$ is the discrete topology on X.

   (b) Suppose that $(X, \mathcal{T})$ is a topological space with the property that the singleton set $\{x\}$ is closed for ever $x \in X$. Must $\mathcal{T}$ be the discrete topology on X?

3. Let X be a set, and $A \subseteq X$ a proper subset. What are the interior and closure (Assignment Problems 3 and 4) of A if X is given

   (a) the discrete topology?

   (b) the indiscrete topology?

4. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

   Find the interior and closure of the subsets

   (a) $\{a, b, c\}$ (b) $\{a, c, d\}$ (c) $\{a, b, d\}$ (d) $\{b\}$ (e) $\{d\}$ (f) $\{b, d\}$

5. (a) Let $(X, \mathcal{T})$ be a topological space, and let $I : X \to X$ be the identity function, defined by $I(x) = x$ for all $x \in X$. Show that I is always continuous when the domain and codomain are given the same topology.

   (b) Now consider $I : \mathbb{R} \to \mathbb{R}$, and investigate whether I is continuous when we allow the domain and codomain to carry different topologies – the answer now depends on the two topologies chosen. Consider (for example) the discrete topology, the indiscrete topology, the cofinite topology, and the Euclidean topology.

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be a subset endowed with the subset topology $\mathcal{T}_S$. Show that a set $C \subseteq S$ is closed if and only if there is some set $D \subseteq X$ that is closed with $C = D \cap S$.

2. Let $(X, \mathcal{T}_X)$ be a topological space, and suppose $X = A \cup B$ for two **closed** subsets $A, B \subseteq X$. Let $(Y, \mathcal{T}_Y)$ be a topological space and $f : X \to Y$ a function. Show that, if $f|_A$ and $f|_B$ are continuous (with respect to the subspace topologies on A and B), then f is continuous.

3. (a) **Definition (Interior of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *interior* of A to be the set

$$\text{Int}(A) = \{\ a \in A \mid \text{there is some neighbourhood } U \text{ of } a \text{ such that } U \subseteq A.\ \}$$

   Prove that $\text{Int}(A)$ is necessarily an open set.

(b) Suppose that $A \subseteq X$ is any subset, and $U \subseteq A$ is an open set. Prove that $U \subseteq \mathrm{Int}(A)$.

4. (a) **Definition (Closure of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *closure* of A to be the set

$$\overline{A} = \{\ x \in X \mid \text{any neighbourhood } U \text{ of } x \text{ contains a point of } A\ \}.$$

Prove that $A \subseteq \overline{A}$.

(b) Prove that $\overline{A}$ is necessarily a closed set.

(c) Suppose that $A \subseteq X$ is any subset, and C is a closed set containing A. Prove that $\overline{A} \subseteq C$.

5. (a) Suppose that $(X, \mathcal{T}_X)$ is a topological space, and that $(Y, \mathcal{T}_Y)$ is a **Hausdorff** topological space. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Suppose that $A \subseteq X$ is a subset such that

$$f(a) = g(a) \qquad \text{for all } a \in A.$$

Prove that

$$f(x) = g(x) \qquad \text{for all } x \in \overline{A}.$$

This says that the values of a continuous function on $\overline{A}$ are completely determined by its values on A.

(b) Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the Euclidean metric, and let Y be a Hausdorff topological space. Let $f : \mathbb{Q} \to Y$ be a function. Briefly explain why there is at most one continuous function $F : \mathbb{R} \to Y$ satisfying $F|_{\mathbb{Q}} = f$. In other words, if f extends to a continuous function $\mathbb{R} \to Y$, then its extension is unique.