## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show that $\mathcal{T}$ is a basis for $\mathcal{T}$.
   (b) Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Show that any collection of open sets in $X$ containing $\mathcal{B}$ is also a basis for $\mathcal{T}$.
2. Verify that the set of open intervals $(a, b) \subseteq \mathbb{R}$ is a basis for $\mathbb{R}$ (with the Euclidean topology).
3. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on $X$
$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$
   (a) Which elements of $X$ are limits of the constant sequence $x_n = d$? The constant sequence $x_n = a$? The constant sequence $x_n = b$?
   (b) Give an example of a sequence in $X$ that does not converge.
4. Let $X = \{0, 1\}$. Find a topology on $X$ for which the following sequence converges:
$$0\,1\,0\,1\,0\,1\,0\,1\cdots$$
5. Suppose that $(X, \mathcal{T})$ is a topological space, and that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $X$ that converges to $a_\infty \in X$. Prove that any subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to $a_\infty \in X$.

## Assignment questions

(Hand these questions in!)

1. Show that the continuous image of a Hausdorff topological space need not be Hausdorff. Specifically, find topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ such that $X$ is Hausdorff but $Y$ is not Hausdorff, and a continuous **surjective** map $f : X \to Y$. Remember to justify your solution!
2. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a basis for $\mathcal{T}$. For a subset $A \subseteq X$, prove the following.
   (a) $\text{Int}(A) = \{a \in A \mid \text{there exists a basis element } B \text{ with } a \in B \subseteq A\}$
   (b) $\overline{A} = \{x \in X \mid \text{any basis element } B \in \mathcal{B} \text{ containing } x \text{ must contain a point of } A\}$
3. **(An extrinsic definition of a basis).** Let $X$ be a set and let $\mathcal{B}$ be a collection of subsets of $X$ such that
   - $\displaystyle\bigcup_{B\in\mathcal{B}} B = X$
   - If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq (B_1 \cap B_2)$.

   Let $\mathcal{T}$ be the collection of subsets of $X$
$$\{\ U \mid U \text{ is a union of elements of } \mathcal{B}\ \}.$$
   Prove that $\mathcal{T}$ is a topology on $X$, and that $\mathcal{B}$ is a basis for $\mathcal{T}$. We say that $\mathcal{T}$ is the topology *generated by* the basis $\mathcal{B}$.

4. Consider the set $\mathbb{R}$ with the **cofinite** topology. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}$.
   (a) Suppose that the sequence has the property that each term is repeated at most finitely many times. More precisely, suppose for each $r \in \mathbb{R}$ that $r = a_n$ for at most finitely many values of $n \in \mathbb{N}$. Which points of $\mathbb{R}$ are limits of the sequence $(a_n)_{n\in\mathbb{N}}$?
   (b) Now suppose the set $\{a_n \mid n \in \mathbb{N}\}$ is finite. Under what conditions will the sequence converge, and what will its limit(s) be?

   Remember to justify your solutions!

5. (a) Consider the following definition.

   **Definition (The product topology).** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the *product topology* $\mathcal{T}_{X\times Y}$ on $X \times Y$ is the collection of subsets of $X \times Y$ generated by the set

   $$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ is open, and } V \subseteq Y \text{ is open}\}.$$

   This means that $\mathcal{T}_{X\times Y}$ consists of all unions of elements of $\mathcal{B}$.

   Verify that $\mathcal{T}_{X\times Y}$ is indeed a topology on $X \times Y$, and that $\mathcal{B}$ is a basis for this topology.
   (b) Prove that the projection map $\pi_X : X \times Y \to X$ is both continuous and open with respect to the topologies $\mathcal{T}_{X\times Y}$ and $\mathcal{T}_X$. (The same argument shows that the projection map $\pi_Y$ is both continuous and open).