Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions.
   (a) Show that if f and g are both surjective, then $g \circ f$ is surjective.
   (b) Show that if f and g are both injective, then $g \circ f$ is injective.
   (c) Show by example that, if we only assume one of f and g is surjective, then $g \circ f$ need not be surjective.
   (d) Show by example that, if we only assume one of f and g is injective, then $g \circ f$ need not be injective.

2. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

   (a) Is X a T_1-space?
   (b) Is X Hausdorff?
   (c) Find the interior, closure, and boundary of $\{a, c, d\}$.
   (d) Find the interior, closure, and boundary of $\{a, b\}$.

3. Consider the following subsets of $\mathbb{R}$
   - $\mathbb{R}$
   - $\varnothing$
   - $\{0, 1\}$
   - $[0, \infty)$
   - $(1, 2)$
   - $[1, 2] \cup [3, \infty)$
   - $(-\infty, 0)$
   - $(-\infty, 0]$
   - $\mathbb{N}$
   - $\{-n \mid n \in \mathbb{N}\}$

   Find the interior, closures, boundaries, and accumulation points of these subsets ...
   (a) ... when $\mathbb{R}$ has the discrete topology.
   (b) ... when $\mathbb{R}$ has the indiscrete topology.
   (c) ... when $\mathbb{R}$ has the Euclidean topology.
   (d) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.
   (e) ... when $\mathbb{R}$ has the cofinite topology.
   (f) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$. .
   (g) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.

Assignment questions

(Hand these questions in!)

1. Let X be a topological space, and $A \subseteq X$. Prove that $\overline{A} = A \cup A'$, where A' is the set of accumulation points of A.

2. (a) Let $A \subseteq B$ be subsets of a topological space X. For each of the following statetments, either give a proof or a counterexample.

(i) $\text{Int}(A) \subseteq \text{Int}(B)$ (ii) $\overline{A} \subseteq \overline{B}$ (iii) $\partial A \subseteq \partial B$

(b) Let A be a subset of a topological space A. Show that $\partial(\text{Int}(A))$ is contained in ∂A, but show by example that these sets need not be equal.

(c) Let A be a subset of a topological space A. Show that $\partial(\overline{A})$ is contained in ∂A, but show by example that these sets need not be equal.

3. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A, B \subseteq X$.

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Show by example that $\overline{A \cap B}$ need not equal $\overline{A} \cap \overline{B}$.

4. Let $g, h : X \to Y$ be continuous maps between topological spaces. Show that if Y is Hausdorff, then the set $S = \{x \in X \mid g(x) = h(x)\}$ is closed.

5. Let $(X, \mathcal{T}_X)$ be a topological space, and endow the product $X \times X$ with the product topology $\mathcal{T}_{X \times X}$. The set

$$\Delta = \{\ (x, x) \mid x \in X\ \} \ \subseteq\ X \times X$$

is called the *diagonal* of $X \times X$. Prove that X is Hausdorff if and only if the diagonal Δ is a closed subset of $X \times X$.