1. **(Quantifier review: order of quantifiers matters!)** Let P be some mathematical assertion. Explain the difference between the statements

   "For all $n \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ such that P holds."

   "There exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, P holds."

   In particular, interpret the following two statements. Re-phrase both statements in colloquial English, and explain why one statement is true and the other is false.

   "For all $n \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ such that $n > N$."

   "There exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > N$."

2. **(Functions review).** Let X and Y be sets, and let $f : X \to Y$ be a function.
   (a) Prove or give a counterexample: If $U, V \subseteq Y$ are disjoint, then $f^{-1}(U)$ and $f^{-1}(V)$ in X are disjoint.
   (b) Prove or give a counterexample: If $U, V \subseteq X$ are disjoint, then $f(U)$ and $f(V)$ in Y are disjoint.

3. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that the following two definitions of convergence are equivalent.

   Definition (Convergence to a_∞). The sequence $(a_n)_{n\in\mathbb{N}}$ converges to a_∞ if for any $\epsilon > 0$, there is some N in $\mathbb{N}$ such that $a_n \in B_\epsilon(x)$ for all $n \geq N$.

   Definition (Convergence to a_∞). The sequence $(a_n)_{n\in\mathbb{N}}$ converges to a_∞ if for any $\epsilon > 0$, there is some $\tilde{N}$ in $\mathbb{N}$ such that $a_n \in B_\epsilon(x)$ for all $n > \tilde{N}$.

4. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Let $S = (0, 1) \subseteq \mathbb{R}$. Give a complete and rigorous proof of what the sets $\mathring{S}$, $\overline{S}$, and ∂S are.

5. Let (X, d) be a metric space. Suppose that $S \subseteq T \subseteq X$.
   (a) Suppose that x is an accumulation point of S. Must x be an accumulation point of T? Give a proof or a counterexample.
   (b) Suppose that x is an accumulation point of T. Must x be an accumulation point of S? Give a proof or a counterexample.

6. Let $A \subseteq \mathbb{R}$ be a nonempty bounded subset. Prove that its supremum $\sup(A)$ is either in the set A, or it is an accumulation point of A.

7. Let (X, d) be a metric space, and let $S \subseteq X$ be a sequentially compact subset.
   (a) Show by example that not every subsequence of a sequence $(a_n)_{n\in\mathbb{N}}$ in S need necessarily converge.
   (b) Prove that no subsequence of $(a_n)_{n\in\mathbb{N}}$ can converge to a point in $X \setminus S$.

8. Find an explicit homeomorphism bewteen the intervals $(0,1)$ and $(1,\infty)$ with the Euclidean metric.

9. Let (X,d) be a metric space with at least two elements. Show that there exist nonempty open sets in X whose closures are disjoint.

10. Let (X,d) be a metric space, and let $S \subseteq X$ be a set with no accumulation points. Prove that S is closed.

11. (a) Let $S \subseteq X$ be a subset of a metric space X. Explain why, to show that S is closed, it suffices to show that S is the preimage of a closed set under a continuous function.
    (b) Show that the following subsets of $\mathbb{R}^2$ (with the usual metric) are closed:

$$\{(x,y) \mid xy = 1\} \qquad S^1 = \{(x,y) \mid x^2+y^2 = 1\} \qquad D^2 = \{(x,y) \mid x^2+y^2 \leq 1\}$$

12. Let (X,d) be a metric space, and let $S \subseteq X$ be a **finite** subset of X. Give a rigorous proof that S . . .

    (a) is closed.
    (b) is bounded.
    (c) has no accumulation points.

13. Let (X,d) be a metric space, and let $S \subseteq X$ be a **finite** subset of X. Is it necessarily true that $\mathring{S} = \varnothing$?

14. Either prove or find a counterexample to each of the following statements.

    (i) Let X be a topological space, and $S \subseteq X$. Then $\overline{S} = X \setminus \text{Int}(X \setminus S)$.
    (ii) Let X be a topological space, and $S \subseteq X$. Then $\overline{S} = \overline{\text{Int}(S)}$.
    (iii) If $A \subseteq B$, then $\text{Int}(A) \subseteq \text{Int}(B)$
    (iv) If $A \subseteq B$, then all accumulation points of A are also accumulation points of B.
    (v) If $\text{Int}(A) = \text{Int}(B)$ and $\overline{A} = \overline{B}$, then $A = B$.
    (vi) If $\text{Int}(A) = \overline{A}$, then A is both open and closed.
    (vii) If A is an open set, then $A \cap \partial A = \varnothing$.
    (viii) Let X be a metric space with the discrete metric, and $A \subseteq X$. Then A has no accumulation points.
    (ix) Let X be a metric space, and $(a_n)_{n\in\mathbb{N}}$ a sequence of points in X converging to a_∞. Then a_∞ is an accumulation point of the set $\{a_n \mid n \in \mathbb{N}\}$.

15. Suppose that (X,d) is a metric space, and that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in X. Show that the sequence of real numbers $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$ (with the Euclidean metric).

16. Suppose that (X,d) is a metric space with the property that every bounded sequence converges. Prove that X is a single point.

17. (a) Prove that $\mathbb{R}$ with the Euclidean metric is a complete metric space.
    (b) Prove that the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ with the Euclidean metric is **not** a complete metric space.
    (c) The function

$$f : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$f(x) = \arctan(x)$$

    is continuous (which you do not need to prove). Use this function to show that the continuous image of a complete metric space need not be complete.
    (d) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, and that X a complete metric space. Suppose that $f : X \to Y$ is a continuous map satisfying

$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

    Show that $f(X)$ is a complete metric space.

18. Let $f : X \to Y$ be a function of metric spaces. Prove that f is open if and only if $f(\text{Int}(A)) \subseteq \text{Int}(f(A))$ for all sets $A \subseteq X$.

19. Let X be a finite set (of, say, n elements), and let d be a metric on X. What is the topology $\mathcal{T}_d$ on X induced by d? Show in particular that this topology will be the same for every possible metric d.

20. Let X be a set, and suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X.
    (a) Show that the intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X.
    (b) Show by example that the union $\mathcal{T}_1 \cup \mathcal{T}_2$ need not be a topology on X.

21. (a) Show that the following collection of subsets of $\mathbb{R}$ forms a topology on $\mathbb{R}$.

$$\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \Big\{(b, \infty) \;\Big|\; b \in \mathbb{R}\Big\}.$$

    (b) Determine whether the topology $\mathcal{T}$ is metrizable.

22. (a) Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space, and let $x_1, \ldots, x_n$ be a finite collection of points in X. Show that there are open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$, and which are pairwise disjoint (this means $U_i \cap U_j = \varnothing$ for all $i \neq j$).
    (b) Let X be a finite topological space. Prove if X is Hausdorff, then it has the discrete topology.