Final Exam

Math 490
18 December 2019
Jenny Wilson

Name: ____________________

Instructions: This exam has 8 questions for a total of 40 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	13	
2	4	
3	4	
4	4	
5	6	
6	2	
7	3	
8	4	
Total:	40	

1. (13 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let X be a metric space, $x \in X$, and $r > 0$. Then any two points y, z in the ball $B_r(x)$ must be distance at most $2r$ apart.

   (b) Let $f : X \to Y$ be a continuous function of metric spaces X and Y. If $(a_n)_{n \in \mathbb{N}}$ is a sequence in X that is Cauchy, then its image $\big(f(a_n)\big)_{n \in \mathbb{N}}$ in Y is also Cauchy.

   (c) Let S be a **finite** subset of a topological space X. Then S has no accumulation points.

   (d) Let (X, d) be a metric space. Then X is T_1, T_2 (Hausdorff), and regular.

   (e) Let X and Y be two non-empty topological spaces with the discrete topology. Then the product topology on $X \times Y$ is the discrete topology.

   (f) Let X be any topological space, and let $\mathbb{R}$ have the standard topology. Then a function $f : X \to \mathbb{R}$ is continuous if and only if $f^{-1}((a, b))$ is open for every open interval $(a, b) \subseteq \mathbb{R}$.

(g) Endow $\mathbb{R}$ and $\mathbb{Q}$ with the topologies induced by the Euclidean metric. Then the only continuous maps $f : \mathbb{R} \to \mathbb{Q}$ are constant maps.

(h) Let X be any topological space, and let $\mathbb{R}$ have the standard topology. Let $f : X \to \mathbb{R}$ be a continuous function, and let $c \in \mathbb{R}$. Then the set $\{x \in X \mid f(x) \leq c\}$ is closed in X.

(i) If A is a subspace of a space X such that $\text{Int}(A)$ is connected, then A is connected.

(j) Consider $\mathbb{R}$ with the topology $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. There is no (continuous) path from $0 \in \mathbb{R}$ to $1 \in \mathbb{R}$.

(k) Let X be a topological space. Then every connected component of X is both open and closed.

(l) Let X be a topological space, and let A, B be a separation of X. Then A is a union of connected components of X, as is B.

(m) Let S be a compact subset of a metric space X. Then S is complete.

2. (4 points) Consider the following statement.

Let $f : X \to Y$ be a continuous function of topological spaces.

If X is __________, then so is $f(X)$.

Circle all properties that truthfully fill in the blank. **No justification needed.**

`metrizable` T_2 `(Hausdorff)` `connected` `disconnected`

`path-connected` `discrete` `compact` `non-compact`

(Here, by "X is discrete" we mean "X has the discrete topology".)

3. (4 points) Consider the following topological spaces X and their subsets S. In each case, compute the interior $\mathrm{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S. **No justification necessary.**

(a) Let $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\Big\}$. Let $S = \{a, b, d\}$.

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

(b) Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{U \mid 0 \in U\} \cup \{\varnothing\}$. Let $S = \{0, 1\}$.

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

4. (4 points) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

   (a) Let $X = \{a, b, c, d\}$ have the topology $\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$.

      (i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$

      (ii) $c,\ d,\ c,\ d,\ c,\ d,\ c,\ d,\ \cdots$

   (b) Let $\mathbb{R}$ have the cofinite topology.

      (ii) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8}, \cdots$

      (iii) $0,\ 1,\ 0,\ 2,\ 0,\ 3,\ 0,\ 4,\ 0,\ 5, \cdots$

5. (6 points) Circle all terms that apply. **No justification necessary.**

   (a) The subspace $(0, 1) \subseteq \mathbb{R}$ with the standard topology is . . .

   `compact` `connected` T_1 T_2 `(Hausdorff)`

   (b) The subspace $\{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ of $\mathbb{R}$ with the standard topology is . . .

   `compact` `connected` T_1 T_2 `(Hausdorff)`

   (c) The topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$ on $\mathbb{R}$ is . . .

   `compact` `connected` T_1 T_2 `(Hausdorff)`

   (d) The topology $\mathcal{T} = \{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$ on $\mathbb{R}$ is . . .

   `compact` `connected` T_1 T_2 `(Hausdorff)`

6. (2 points) For each of the following maps f, circle all properties that apply.

(a) $f : (\mathbb{R}, \text{ Euclidean}) \to (\mathbb{R}, \text{ cofinite})$
$$f(x) = x$$
`continuous` `open`

(b) Let $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$.
$$f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$$
$$f(x) = x + 1$$
`continuous` `open`

7. (3 points) Let X_1 be a topological space with basis $\mathcal{B}_1$, and let X_2 be a topological space with basis $\mathcal{B}_2$. Show that the set

$$\mathcal{B} = \{ \ B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2 \ \}$$

is a basis for the product topology $\mathcal{T}_{X_1 \times X_2}$.

8. (4 points) Show that a topological space X is Hausdorff if and only if, for each $x \in X$,

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \quad = \quad \{x\}.$$

Blank page for extra work.