Final Exam

Math 490
18 December 2019
Jenny Wilson

Name: ______________________________

Instructions: This exam has 8 questions for a total of 40 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	13	
2	4	
3	4	
4	4	
5	6	
6	2	
7	3	
8	4	
Total:	40	

1. (13 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let X be a metric space, $x \in X$, and $r > 0$. Then any two points y, z in the ball $B_r(x)$ must be distance at most $2r$ apart.

   True. *Hint:* Use the triangle inequality.

   (b) Let $f : X \to Y$ be a continuous function of metric spaces X and Y. If $(a_n)_{n \in \mathbb{N}}$ is a sequence in X that is Cauchy, then its image $\big(f(a_n)\big)_{n \in \mathbb{N}}$ in Y is also Cauchy.

   False. Let $X = Y = (0, \infty)$ with the Euclidean metric. Consider the continuous function $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x}$. Then the sequence $a_n = \frac{1}{n}$ is Cauchy in X, but its image $f(a_n) = n$ is not Cauchy.

   (c) Let S be a **finite** subset of a topological space X. Then S has no accumulation points.

   False. Consider $X = S = \{0, 1\}$ with the indiscrete topology. Then both 0 and 1 are accumulation points of S.

   (d) Let (X, d) be a metric space. Then X is T_1, T_2 (Hausdorff), and regular.

   True. See Worksheet #4 Problem 2(a), Homework #6 Problem 5, and Homework #11 Problem 5(c).

   (e) Let X and Y be two non-empty topological spaces with the discrete topology. Then the product topology on $X \times Y$ is the discrete topology.

   True. *Hint:* It is enough to verify that the singleton set $\{(x, y)\}$ is open for each point $(x, y) \in X \times Y$. Observe that $\{(x, y)\} = \{x\} \times \{y\}$, where $\{x\} \subseteq X$ and $\{y\} \subseteq Y$ are open.

   (f) Let X be any topological space, and let $\mathbb{R}$ have the standard topology. Then a function $f : X \to \mathbb{R}$ is continuous if and only if $f^{-1}((a, b))$ is open for every open interval $(a, b) \subseteq \mathbb{R}$.

   True. *Hint:* The open intervals $\{(a, b) \mid a, b \in \mathbb{R}\}$ form a basis for the standard topology on $\mathbb{R}$. The statement then follows from Worksheet #11 Problem 4.

(g) Endow $\mathbb{R}$ and $\mathbb{Q}$ with the topologies induced by the Euclidean metric. Then the only continuous maps $f : \mathbb{R} \to \mathbb{Q}$ are constant maps.

True. *Hint:* Homework #10 Problem 2 implies that $\mathbb{R}$ is connected, and moreover that the continuous image of a connected set is connected. By Homework #11 Problem 1(d), the maximal connected subsets of $\mathbb{Q}$ are single points $\{q\}$, so the image of f must be a point.

(h) Let X be any topological space, and let $\mathbb{R}$ have the standard topology. Let $f : X \to \mathbb{R}$ be a continuous function, and let $c \in \mathbb{R}$. Then the set $\{x \in X \mid f(x) \leq c\}$ is closed in X.

True. *Hint:* This set is exactly $f^{-1}((-\infty, c])$. The preimage of a closed set under a continuous function is closed.

(i) If A is a subspace of a space X such that $\text{Int}(A)$ is connected, then A is connected.

False. Consider $A = (0, 1) \cup \{2\}$ as a subspace of $X = \mathbb{R}$ with the standard topology. Then A is disconnected, but $\text{Int}(A) = (0, 1)$ is connected.

(j) Consider $\mathbb{R}$ with the topology $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. There is no (continuous) path from $0 \in \mathbb{R}$ to $1 \in \mathbb{R}$.

False. The function $f : [0, 1] \to \mathbb{R}$ given by $f(t) = t$ defines a path from 0 to 1.

(k) Let X be a topological space. Then every connected component of X is both open and closed.

False. For example, the connected components of $\mathbb{Q}$ are singleton sets $\{x\}$ (Homework #11 Problem 1(d)), which are closed but not open.

(l) Let X be a topological space, and let A, B be a separation of X. Then A is a union of connected components of X, as is B.

True. *Hint:* Worksheet #14 Problem 5 states that every connected subset of X must be contained in either A or B.

(m) Let S be a compact subset of a metric space X. Then S is complete.

True. *Hint:* By Homework #11 Problem 4, the subset S is compact if and only if it is sequentially compact. Worksheet #8 Problem 1 states that sequentially compact spaces are complete.

2. (4 points) Consider the following statement.

   Let $f : X \to Y$ be a continuous function of topological spaces.

   If X is __________, then so is $f(X)$.

   Circle all properties that truthfully fill in the blank. **No justification needed.**

   `metrizable` T_2 `(Hausdorff)` `connected` `disconnected`

   `path-connected` `discrete` `compact` `non-compact`

   (Here, by "X is discrete" we mean "X has the discrete topology".)

3. (4 points) Consider the following topological spaces X and their subsets S. In each case, compute the interior $\text{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S. **No justification necessary.**

   (a) Let $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\Big\}$. Let $S = \{a, b, d\}$.

   $\text{Int}(S)$: $\{a, b\}$ $\overline{S}$: $\{a, b, c, d\}$ ∂S: $\{c, d\}$ S': $\{c, d\}$

   (b) Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{U \mid 0 \in U\} \cup \{\varnothing\}$. Let $S = \{0, 1\}$.

   $\text{Int}(S)$: $\{0, 1\}$ $\overline{S}$: $\mathbb{R}$ ∂S: $\mathbb{R}\setminus\{0,1\}$ S': $\mathbb{R}\setminus\{0\}$

4. (4 points) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

(a) Let $X = \{a, b, c, d\}$ have the topology $\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$.

(i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$ limits: $\{b, c, d\}$

(ii) $c,\ d,\ c,\ d,\ c,\ d,\ c,\ d,\ \cdots$ Does not converge.

(b) Let $\mathbb{R}$ have the cofinite topology.

(ii) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8}, \cdots$ limits: $\mathbb{R}$

(iii) $0,\ 1,\ 0,\ 2,\ 0,\ 3,\ 0,\ 4,\ 0,\ 5, \cdots$ limits: $\{0\}$

5. (6 points) Circle all terms that apply. **No justification necessary.**

(a) The subspace $(0, 1) \subseteq \mathbb{R}$ with the standard topology is ...

`compact` **`connected`** **T_1** **T_2 `(Hausdorff)`**

(b) The subspace $\{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ of $\mathbb{R}$ with the standard topology is ...

`compact` `connected` **T_1** **T_2 `(Hausdorff)`**

(c) The topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$ on $\mathbb{R}$ is ...

`compact` **`connected`** T_1 T_2 `(Hausdorff)`

(d) The topology $\mathcal{T} = \{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$ on $\mathbb{R}$ is ...

`compact` **`connected`** T_1 T_2 `(Hausdorff)`

6. (2 points) For each of the following maps f, circle all properties that apply.

(a) $f : (\mathbb{R},\ \text{Euclidean}) \to (\mathbb{R},\ \text{cofinite})$

$f(x) = x$

`continuous` `open`

(b) Let $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$.

$f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$

$f(x) = x + 1$

`continuous` `open`

7. (3 points) Let X_1 be a topological space with basis $\mathcal{B}_1$, and let X_2 be a topological space with basis $\mathcal{B}_2$. Show that the set

$$\mathcal{B} = \{\ B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\ \}$$

is a basis for the product topology $\mathcal{T}_{X_1 \times X_2}$.

Solution. To verify that $\mathcal{B}$ is a basis for the topology on $X_1 \times X_2$, by the basis criterion (Worksheet #11 Problem 1), it suffices to check (i) that the elements of $\mathcal{B}$ are open, and (ii) that for every open set $W \subseteq X_1 \times X_2$, and every point $(x_1, x_2) \in W$, there is some basis element $B \in \mathcal{B}$ with $(x_1, x_2) \in B \subseteq W$.

The topology on $X_1 \times X_2$ is defined by a basis of open sets $V_1 \times V_2$ where $V_1 \subseteq X_1$ and $V_2 \subseteq X_2$ are open sets. Since any basis elements $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ are open by definition of a basis, $B_1 \times B_2$ is an open subset of $X_1 \times X_2$.

Now, let W be any open subset of $X_1 \times X_2$, and let $(x_1, x_2) \in W$. By definition of the product topology, there must be a pair of open subsets $V_1 \subseteq X_1$ and $V_2 \subseteq X_2$ such that $(x_1, x_2) \in V_1 \times V_2 \subseteq W$. This means $x_1 \in V_1$ and $x_2 \in V_2$. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for X_1 and X_2, respectively, the basis criterion implies that there are basis elements $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ satisfying $x_1 \in B_1 \subseteq V_1$ and $x_2 \in B_2 \subseteq V_2$.

But then $(x_1, x_2) \in B_1 \times B_2 \subseteq V_1 \times V_2 \subseteq W$, and we conclude by the basis criterion that $\mathcal{B}$ is a basis, as claimed.

8. (4 points) Show that a topological space X is Hausdorff if and only if, for each $x \in X$,

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} = \{x\}.$$

Solution. Since $x \in U \subseteq \overline{U}$ for all neighbourhoods U of x, it follows that

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \supseteq \{x\}$$

for any topological space X and point $x \in X$. Hence, our goal is to show that

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \subseteq \{x\}$$

if and only if X is Hausdorff.

First suppose that X is Hausdorff. This means, for any $x, y \in X$ with $y \neq x$, there are disjoint neighbourhoods U_x of x and U_y of y. But, since y has a neighbourhood that does not intersect U_x, we can conclude that $y \notin \overline{U_x}$. Hence for all $y \neq x$,

$$y \notin \bigcap_{U \text{ a neighbourhood of } x} \overline{U}.$$

We conclude that, if X is Hausdorff, then for any $x \in X$,

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \subseteq \{x\}.$$

Next suppose that

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \subseteq \{x\}$$

for all $x \in X$, and consider any pair of distinct points $x, y \in X$. By assumption, there must be some neighbourhood U_x of x so that $y \notin \overline{U_x}$. This means that there is some neighbourhood U_y of y that does not intersect U_x. Thus x and y have disjoint neighbourhoods U_x and U_y, and we conclude that X is Hausdorff as desired.

Blank page for extra work.