Midterm Exam

Math 490
22 October 2019
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Name: ______________________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	9	
2	4	
3	3	
4	4	
Total:	20	

1. (9 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let $f : X \to Y$ be a continuous function of metric spaces X and Y. Then for any $A \subseteq Y$, the preimage $f^{-1}(\overline{A}) \subseteq X$ is closed.

   Solution: True. You proved that $\overline{A}$ is always closed, and that the preimage of a closed set under a continuous function is closed.

   (b) Let X be a metric space, and $S \subseteq X$. Then $\partial S = \partial(X \setminus S)$.

   Solution: True. See Homework #4 Problem 3(d).

   (c) Let X be a metric space, and $S \subseteq X$. Then $\partial S = \partial(\overline{S})$.

   Solution: False. Consider $S = \mathbb{Q}$ in $X = \mathbb{R}$ with the Euclidean metric. Then $\partial S = \partial \mathbb{Q} = \mathbb{R}$ but $\partial(\overline{S}) = \partial \mathbb{R} = \varnothing$.

   (d) Let X be a metric space, and $A \subseteq X$. Then a point $x \in X$ is contained in $\overline{A}$ if and only if x is an accumulation point of A.

   Solution: False. Consider $X = \mathbb{R}$ with the Euclidean metric, and $S = \{1\}$. Then $1 \in \overline{S}$, but 1 is not an accumulation point of S. (By Quiz #3, finite sets have no accumulation points).

   (e) Let X and Y be metric spaces, and let $f : X \to Y$ be a continuous, invertible, open map. Then f is a homeomorphism.

   Solution: True. To be a homeomorphism, the map f must be continuous, invertible, and have a continuous inverse f^{-1}, and so we only need to check only the third condition. Let $U \in X$ be open. But then its preimage $(f^{-1})^{-1}(U) = f(U)$ is open by assumption that f is an open map, so f^{-1} is continuous.

(f) Every metric space is Hausdorff.

Solution: True. See Worksheet #4 Problem 2(a).

(g) Let X, Y be metric spaces, and $f : X \to Y$ a continuous function. If $S \subseteq Y$ is sequentially compact, then $f^{-1}(S)$ is sequentially compact.

Solution: False. Consider $X = Y = \mathbb{R}$ with the Euclidean metric, and let f be the constant map $f(x) = 0$. Then f is continuous, and $S = \{0\} \subseteq \mathbb{R}$ is sequentially compact, but $f^{-1}(\{0\}) = \mathbb{R}$ is not sequentially compact.

(h) Every sequentially compact metric space is complete.

Solution: True. See Worksheet #8 Problem 1.

(i) Every complete metric space is sequentially compact.

Solution: False. For example, we proved on Worksheet #8 that $\mathbb{R}$ is complete, but it cannot be sequentially compact, since it is not bounded.

2. (4 points) Below are two metric spaces X and subsets A. For each subset, state the interior, closure, and boundary of A, and its set A' of accumulation points. **No justification needed.**

$X = \mathbb{R}$ with the Euclidean metric, $A = \left\{ \frac{(-1)^n}{n} \;\middle|\; n \in \mathbb{N} \right\}$.

$\mathrm{Int}(A) = \underline{\quad \varnothing \quad}$ $\quad \overline{A} = \underline{\quad A \cup \{0\} \quad}$ $\quad \partial A = \underline{\quad A \cup \{0\} \quad}$ $\quad A' = \underline{\quad \{0\} \quad}$

$X = \mathbb{R}$ with the discrete metric, $A = (0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

$\mathrm{Int}(A) = \underline{\quad A \quad}$ $\quad \overline{A} = \underline{\quad A \quad}$ $\quad \partial A = \underline{\quad \varnothing \quad}$ $\quad A' = \underline{\quad \varnothing \quad}$

3. (3 points) Let (X, d) be a metric space, and let $A \subseteq X$. Prove that, if x is an accumulation point of A, then every neighbourhood U of x contains infinitely many points of A.

Solution 1. We will prove the contrapostive: suppose that a point $x \in X$ has some neighbourhood U that contains only finitely many points of A. We will show that x is not an accumulation point of A. To do this, we must find a ball $B_r(x)$ that contains no points of A except perhaps x itself.

Since U is open, there must be some ball $B_\epsilon(x)$ around x contained in U. By assumption, this ball $B_\epsilon(x)$ must contain only finitely many points in A. Let

$$r = \min_{\substack{a \in A \cap B_\epsilon(x) \\ a \neq x}} \{\epsilon, d(x, a)\}.$$

Since r is the minimum of a finite list of strictly positive numbers, we know $r > 0$.

We may therefore consider the ball $B_r(x)$. Since $r \leq \epsilon$, $B_r(x) \subseteq B_\epsilon(x) \subseteq U$. But since $r \leq d(x, a)$ for all $a \in A \cap B_\epsilon(x)$ disctinct from x , it follows that $B_r(x)$ does not contain any points of A, except perhaps x itself. We conclude that x is not an accumulation point of A, as claimed.

Solution 2. Let x be an accumulation poinit of A, and let U be any neighbourhood of x. We will identify an infinite set of elements of A in U.

Since x is an interior point of U, there is some ball $B_\epsilon(x) \subseteq U$. By definition of accumulation point, this ball must contain some point $a_1 \in A$ with $a_1 \neq x$.

Since $a_1 \neq x$, the distance $d(a_1, x)$ must be strictly positive. Hence we can consider the ball $B_{d(a_1,x)}(x) \subseteq B_\epsilon(x) \subseteq U$. By definition of accumulation point, this ball must contain some point $a_2 \in A$ distinct from x. Moreover, by construction, $a_1 \notin B_{d(a_1,x)}(x)$, so $a_2 \neq a_1$.

We iterate this process. In general, given a point $a_n \in A \cap B_\epsilon(x)$ distinct from x, we can find a new point a_{n+1} in $B_{d(x,a_n)}(x)$ that is contained in A and distinct from x. Hence we have constructed a sequence $(a_n)_{n \in \mathbb{N}}$ of distinct points of A contained in U. We conclude that U must contain infinitely many points of A, as claimed.

4. (4 points) Let $(X_1, d_1), (X_2, d_2), (Y_1, D_1), (Y_2, D_2)$ be metric spaces. Let $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ be continuous functions. Prove that the function

$$(f \times g) : X_1 \times X_2 \to Y_1 \times Y_2$$
$$(f \times g)(x_1, x_2) = \Big(f(x_1), g(x_2) \Big)$$

is continuous with respect to the product metrics on $X_1 \times X_2$ and $Y_1 \times Y_2$.

Solution. To prove that $(f \times g)$ is continuous, we will verify that the preimage of an open subset of $Y_1 \times Y_2$ is open in $X_1 \times X_2$.

Let $U \subseteq Y_1 \times Y_2$ be open, and consider a point (x_1, x_2) in its preimage $(f \times g)^{-1}(U)$. To prove that $(f \times g)^{-1}(U)$ is open, by Worksheet #9 Problem 3, it is enough to show that the point (x_1, x_2) has an open neighbourhood contained in $(f \times g)^{-1}(U)$.

Since $(x_1, x_2) \in (f \times g)^{-1}(U)$, by definition, this means that the point $(f \times g)(x_1, x_2) \in U$. But we proved on Worksheet #7 that there is then some neighbourhood $U_1 \times U_2$ that contains $(f \times g)(x_1, x_2)$ and is contained in U, with $U_1 \subseteq Y_1$ and $U_2 \subseteq Y_2$ open sets.

Now

$$\begin{aligned}
(f \times g)^{-1}(U_1 \times U_2) &= \{(a_1, a_2) \mid (f \times g)(a_1, a_2) \in U_1 \times U_2\} \\
&= \{(a_1, a_2) \mid \Big(f(a_1), g(a_2) \Big) \in U_1 \times U_2\} \\
&= \{(a_1, a_2) \mid f(a_1) \in U_1 \text{ and } g(a_2) \in U_2\} \\
&= \{(a_1, a_2) \mid a_1 \in f^{-1}(U_1) \text{ and } a_2 \in g^{-1}(U_2)\} \\
&= f^{-1}(U_1) \times g^{-1}(U_2).
\end{aligned}$$

But the subsets $f^{-1}(U_1) \subseteq X_1$ and $g^{-1}(U_2) \subseteq X_2$ must be open, since f and g are both continuous by assumption. We proved on Worksheet #7 that the set $f^{-1}(U_1) \times g^{-1}(U_2)$ must therefore be open.

Then (x_1, x_2) is contained in the open set $f^{-1}(U_1) \times g^{-1}(U_2)$, and since $U_1 \times U_2 \subseteq U$, we find that

$$f^{-1}(U_1) \times g^{-1}(U_2) = (f \times g)^{-1}(U_1 \times U_2) \subseteq (f \times g)^{-1}(U).$$

By Worksheet #9 Problem 3, we conclude that the set $(f \times g)^{-1}(U)$ is open, and hence that $(f \times g)$ is continuous.