Name: ______________________________ Score (Out of 6 points):

1. (4 points) For each of the following statements: if the statement is always true, write "True". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "False" without a counterexample.

   (a) Let A be a subset of a topological space X. If A is connected, then $\overline{A}$ is connected.

   True. See Homework #10 Problem 1.

   (b) Let A be a subset of a topological space X. If $\overline{A}$ is connected, then A is connected.

   False. For example, consider the subset $A = (0,1) \cup (1,2) \subseteq \mathbb{R}$. The set A is disconnected, but its closure $\overline{A} = [0,2]$ is connected.

   (c) Let X be a topological space with basis $\mathcal{B}$. If X is disconnected, then there exist basis elements A, B in $\mathcal{B}$ that are a separation of X.

   False. For example, let $X = (-\infty, 0) \cup (0, \infty)$ with the Euclidean metric, and let $\mathcal{B}$ be the corresponding basis of open balls. Then the union of any two basis elements $A, B \subseteq \mathcal{B}$ must be a bounded subset of X, so cannot cover X.

   (d) Any continuous function from $\mathbb{R}$ (with the standard topology) to a discrete space X must be a constant function.

   True. *Hint:* You proved on Homework #10 Problem 2 that $\mathbb{R}$ is connected, and moreover that the continuous image of a connected set is connected. The only nonempty connected subsets of a discrete space X are singleton sets, so the image of a continuous map must be a point.

2. (2 points) Let $X = \{a, b, c, d\}$ be a topological space with the topology

$$\mathcal{T} = \Big\{\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}\Big\}.$$

Write down a formula for a continuous path in X from a to d. **No justification necessary.**

We seek a function $\gamma : [0,1] \to X$ with the property that

$$\gamma(0) = a, \qquad \gamma(1) = d, \qquad \gamma^{-1}(U) \text{ is open for each open subset } U \in \mathcal{T}.$$

One such function is the following:

$$\gamma : [0,1] \longrightarrow X$$
$$\gamma(t) = \begin{cases} a, & t \in [0, \frac{1}{2}), \\ d, & t \in [\frac{1}{2}, 1] \end{cases}$$

Then

$$\gamma^{-1}(\varnothing) = \gamma^{-1}(\{b\}) = \varnothing,$$
$$\gamma^{-1}(\{a\}) = \gamma^{-1}(\{a,b\}) = \gamma^{-1}(\{a,b,c\}) = \left[0, \frac{1}{2}\right),$$
$$\gamma^{-1}(\{a,b,d\}) = \gamma^{-1}(\{a,b,c,d\}) = [0,1].$$

These are all open subsets of $[0,1]$, so the function γ is continuous as desired.