Name: ______________________________ Score (Out of 6 points):

1. (3 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If $C \subseteq Y$ is compact, then $f^{-1}(C)$ is compact.

   False. For example, consider $\mathbb{R}$ with the standard topology, and consider the constant map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for all x. Then f is continuous, and $\{0\} \subseteq \mathbb{R}$ is finite and therefore compact, but $f^{-1}(\{0\}) = \mathbb{R}$ is unbounded and therefore noncompact.

   (b) Let $(X, \mathcal{T})$ be a compact topological space. Then every closed subset of X is compact.

   True. See Worksheet #16 Problem 3.

   (c) Let $(X, \mathcal{T})$ be a compact topological space. Then every compact subset of X is closed in X.

   False. For example, consider the set $X = \{1, 2\}$ with the indiscrete metric. The subset $\{1\}$ is finite and therefore compact, but $\{1\}$ is not closed.

2. (3 points) Let X be a compact space and Y a Hausdorff space. Suppose that a map $f : X \to Y$ is continuous, and has an inverse f^{-1}. Prove that f^{-1} is continuous.

   This result proves:

   > **Theorem.** A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

   Solution. Let $f : X \to Y$ be a continuous map with inverse $f^{-1} : Y \to X$. To show that f^{-1} is continuous, it suffices to show that $(f^{-1})^{-1}(C)$ is closed in Y for every closed subset C of X.

   We first observe that

$$\begin{aligned}(f^{-1})^{-1}(C) &= \{y \in Y \mid f^{-1}(y) \in C\} \\ &= \{y \in Y \mid f^{-1}(y) = c \text{ for some } c \in C\} \\ &= \{y \in Y \mid y = f(c) \text{ for some } c \in C\} \\ &= f(C),\end{aligned}$$

   so our goal is to show that $f(C)$ is closed in Y for every closed subset $C \subseteq X$.

   Let C be a closed subset of X. Since X is compact, by Worksheet #16 Problem 3, it follows that C is compact. Since f is continuous, by Worksheet #16 Problem 2, it follows that $f(C)$ is a compact subset of Y. But since Y is Hausdorff, it follows from Worksheet #16 Problem 4(b) that $f(C)$ is closed in Y.

   Thus $(f^{-1})^{-1}(C) = f(C)$ is a closed subset of Y, and we conclude that f^{-1} is continuous as claimed.