Name: ______________________________ Score (Out of 6 points):

1. (4 points) Fix $n \in \mathbb{N}$, and let $X = \{0, 1\}^n$, the set of ordered n-tuples of 0's and 1's. We define the *Hamming distance* on X to be the function $d : X \times X \to \mathbb{R}$, where $d(x, y)$ is defined to be the number of coordinates where the n-tuples x and y differ. For example, if $n = 5$, the elements $(0, 0, 1, 0, 0)$ and $(0, 0, 1, 1, 0)$ have Hamming distance 1, because they differ in just one coordinate, the fourth, and agree elsewhere. Hamming distance is an important concept in coding theory.

Prove that the Hamming distance d is a metric on X. Your solution should involve a complete statement of the definition of a metric.

To verify that d is a metric, we must establish that the three conditions hold:

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

To verify (M1), we note that by definition, $d(x, y)$ is always a nonnegative integer, and $d(x, y) = 0$ if and only if x and y agree in every coordinate, that is, $x = y$.

Condition (M2) follows because $d(x, y)$ is by its definition symmetric in x and y: if x differs from y in m positions, then y differs from x in m positions.

Finally, we check the triangle inequality (M3). Let $x, y, z \in X$. Let $d(x, y) = m$ and $d(y, z) = p$. Suppose that x and y differ in the m coordinate positions $i_1, i_2, \ldots, i_m$, and that y and z differ in the p coordinate positions $j_1, j_2, \ldots j_p$. Then x and z can only possibly differ in coordinate positions $i_1, i_2, \ldots, i_m, j_1, j_2, \ldots j_p$, and this list has at most $p + m$ distinct entries. Thus

$$d(x, z) \leq p + m = d(x, y) + d(y, z).$$

2. (2 points) Let (X, d) be a metric space, and let $x, y \in X$ be distinct points. Prove that there exists an open ball in X that contains x but not y.

   Remark: This result shows that metric spaces are T_1-*spaces*.

   Let x and y be distinct points of X. Then, by definition of a metric, the distance

   $$d(x, y) > 0.$$

   Let $r = \frac{1}{2}d(x, y)$. By construction $r > 0$. Let

   $$U = B_r(x) = \{\tilde{x} \in X \mid d(x, \tilde{x}) < r\}.$$

   Since $d(x, x) = 0 < r$, it follows that $x \in U$. However, because

   $$d(x, y) = 2r > r,$$

   the point y is not in U. This proves the claim.