Name: ______________________ Score (Out of 8 points):

1. (2 points) Let (X, d_X) and (Y, d_Y) be metric spaces. State what it means for a function $f : X \to Y$ to be continuous. You may use any of our equivalent definitions.

Solution: Here are some of our equivalent definitions.

A map $f : X \to Y$ is *continuous at* $x \in X$ if, for every $\epsilon > 0$, there is some $\delta > 0$ with the property that whenever $\tilde{x} \in X$ satisfies $d_X(x, \tilde{x}) < \delta$, then $d_Y(f(x), f(\tilde{x})) < \epsilon$. The map f is *continuous* if it is continuous at every point $x \in X$.

A map $f : X \to Y$ is *continuous at* $x \in X$ if, for every $\epsilon > 0$, there is some $\delta > 0$ so that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$. The map f is *continuous* if it is continuous at every point $x \in X$.

A map $f : X \to Y$ is *continuous* if, for every open subset $U \subseteq Y$, the preimage $f^{-1}(U)$ is open in X.

A map $f : X \to Y$ is *continuous* if, for every closed subset $C \subseteq Y$, the preimage $f^{-1}(C)$ is a closed subset of X.

2. (3 points) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $y_0 \in Y$. Show that the constant function

$$\begin{aligned} f : X &\longrightarrow Y \\ f(x) &= y_0 \qquad \text{for all } x \end{aligned}$$

is always continuous.

Solution: We will use the following characterization of continuity: the function f is continuous if and only if the preimage of every open subset of Y is open in X.

So let U be any open subset of Y. Then the preimage of U under f is given by

$$f^{-1}(U) = \begin{cases} X, & y_0 \in U \\ \varnothing, & y_0 \notin U \end{cases}.$$

We proved that the both sets X and $\varnothing$ are open in X. Hence, the preimage of an open set $U \subseteq Y$ is always open, and we conclude that the constant function f is continuous.

3. (3 points) Let (X, d_X) and (Y, d_Y) be metric spaces. Let S be a subset of X. Recall that S inherits a metric space structure under restriction of the metric on X.

   Let $f : X \to Y$ be a function. Recall that the *restriction* of f to S, often written $f|_S$, is the function

$$f|_S : S \longrightarrow Y$$
$$f|_S(s) = f(s).$$

   Prove that, if f is a continuous function, then its restriction $f|_S$ to S is also a continuous function.

Solution: We will use the epsilon-delta characterization of continuity. To show that $f|_S$ is continuous, we must show that it is continuous at every point $s \in S$.

So fix $s_0 \in S$, and let $\epsilon > 0$. Since f is continuous at s_0, there exists some $\delta > 0$ so that for any $x \in X$ with $d_X(x, s_0) < \delta$, it follows that $d_Y(f(x), f(s_0)) < \epsilon$. But then in particular, for any $s \in S \subseteq X$ with $d_X(s, s_0) < \delta$, it follows that

$$d_Y(f|_S(s), f|_S(s_0)) = d_Y(f(s), f(s_0)) < \epsilon.$$

Hence, by definition, the function $f|_S : S \to Y$ is continuous at s_0. Since s_0 was arbitrary, we conclude that $f|_S$ is continuous on S.