Name: ______________________________ Score (Out of 8 points):

1. (2 points) Let (X, d) be a metrc space, and let $A \subseteq X$. Define an *accumulation point* of A.

   Solution: A point $x \in X$ is an *accumulation point* of A if, for every $r > 0$, the ball $B_r(x)$ contains a point of A distinct from x.

2. (3 points) Show that a **finite** subset A of a metric space (X, d) has no accumulation points.

   Solution: Let x in X. To show that x is not an accumulation point of A, we must show that there exists some $r > 0$ so that the ball $B_r(x)$ contains no points of A, except perhaps x itself. Let

   $$r = \min_{a \in A, a \neq x} d(x, a).$$

   Since r is the minimum of a finite set of strictly positive numbers, r must itself be strictly positive. Moreover, by construction, $d(x, a) \geq r$ for all $a \in A$ with $a \neq x$, so $a \notin B_r(x)$ for all $a \in A$ with $a \neq x$. Thus $B_r(x)$ is the desired ball, and we conclude that x is not an accumulation point of A. Since x was arbitrary, we see that A has no accumulation points.

3. (3 points) Let (X, d) be a metric space, and let $A \subseteq X$. Show that x is an accumulation point of A if and only if the following property holds: every open neighbourhood U of x contains a point of A distinct from x.

Solution: We first assume that x has the property that all of its open neighbourhoods contain a point of A distinct from x. In particular, this means that for every $r > 0$, the open neighbourhood $B_r(x)$ of x contains a point of A distinct from x. Thus x is an accumulation point of A.

Conversely, assume that x is an accumulation point of A. Let U be a neighbourhood of x; we must show that U contains a point of A distinct from x. Since U is open, x must be an interior point of U, so there exists some $r > 0$ so that $B_r(x) \subseteq U$. Since x is an accumulation point of A, there is some point a of A contained in $B_r(x)$ with $a \neq x$. But then a is contained in U, and we have found the desired point.