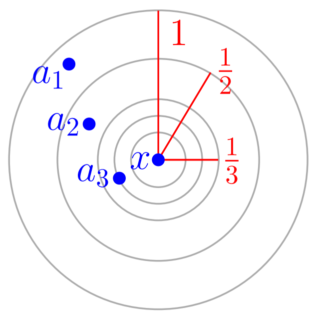
Name: ______________________________ Score (Out of 5 points):

1. (4 points) Let (X, d) be a metric space, and let $A \subseteq X$. Show that, if $x \in \overline{A}$, then there exists a sequence of points in A converging to x.

Solution: Let $x \in \overline{A}$. This means, by definition of closure, that for every $r > 0$, the open ball $B_r(x)$ contains a point $a \in A$.

Construct a sequence as follows. Let a_1 be a point of A in the ball $B_1(x)$. Let a_2 be a point of A in the ball $B_{\frac{1}{2}}(x)$. In general, for $n \in \mathbb{N}$, let a_n be any point of A in the ball $B_{\frac{1}{n}}(x)$.



The resultant sequence $(a_n)_{n \in \mathbb{N}}$ satisfies $\{a_n\}_{n \in \mathbb{N}} \subseteq A$ by construction. We will prove that it converges to the point x.

Let $\epsilon > 0$. Choose N large enough so that $N > \frac{1}{\epsilon}$, equivalently, $\frac{1}{N} < \epsilon$. Then for all $n \geq N$, we find that $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, and

$$a_n \in B_{\frac{1}{n}}(x) \subseteq B_\epsilon(x).$$

Thus $\lim_{n \to \infty} a_n = x$, which concludes the proof.

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2. (1 point) If the following statement is true, write `True`. Otherwise, state a counterexample. No justification needed.

   Let X be a metric space, and let $A, B \subseteq X$. Then $\text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B)$.

   Solution: This statement is false. Consider $X = \mathbb{R}$ with the Euclidean metric, and the subsets $A = [0,1]$ and $B = [1,2]$. Then

   $$\text{Int}\Big([0,1] \cup [1,0]\Big) = \text{Int}\Big([0,2]\Big) = (0,2) \qquad \text{but} \qquad \text{Int}\Big([0,1]\Big) \cup \text{Int}\Big([1,2]\Big) = (0,1) \cup (1,2).$$

   The first set contains 1 and the second does not.