Name: ____________________ Score (Out of 6 points):

1. (3 points) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous map. Prove that if $S \subseteq X$ is sequentially compact, then $f(S) \subseteq Y$ is sequentially compact.

Solution: Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of points in $f(S)$. To prove that $f(S) \subseteq Y$ is sequentially compact, we must find a subsequence that converges to a point in $f(S)$.

For each n the point y_n is in the image of S under f, so we can find a point s_n in S with $f(s_n) = y_n$. But S is sequentially compact by assumption, so the resulting sequence $(s_n)_{n\in\mathbb{N}}$ has a subsequence $(s_{n_i})_{i\in\mathbb{N}}$ that converges to some point $s \in S$.

We will consider the corresponding subsequence $(y_{n_i})_{i\in\mathbb{N}}$. By Homework #3 Problem 4, we know that since f is continuous,

$$\lim_{i\to\infty} y_{n_i} = \lim_{i\to\infty} f(s_{n_i}) = f\left(\lim_{i\to\infty} s_{n_i}\right) = f(s).$$

Thus the subsequence $(y_{n_i})_{i\in\mathbb{N}}$ converges to the point $f(s) \in f(S)$, and we conclude that $f(S)$ is sequentially compact.

2. (3 points) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous map. Either prove the following statement, or construct (with justification) a counterexample: If $B \subseteq X$ is a bounded subset, then its image $f(B) \subseteq Y$ is bounded.

Solution: This statement is false. Consider, for example, the continuous function

$$f : \left(\frac{-\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R}$$
$$f(x) = \tan(x)$$

where both $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbb{R}$ have the standard Euclidean metric.

Then $B = X = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is bounded, since $B = B_{\frac{\pi}{2}}(0)$. But its image $f(B) = \mathbb{R}$ is not bounded, since $\mathbb{R}$ is not contained in the ball $B_R(a) = (a - R, a + R)$ for any $a \in \mathbb{R}$ or $R > 0$.