Name: ____________________________________ Score (Out of 7 points):

1. (3 points) Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be a subset endowed with the subspace topology $\mathcal{T}_S$. Suppose that X is Hausdorff. Show that S is Hausdorff.

Solution: Recall that, by definition, the subspace topology on S is the topology

$$\mathcal{T}_S = \{U \cap S \mid U \in \mathcal{T}_X\}.$$

The statement that X is Hausdorff means that, for every pair of distinct points $x, y \in X$, there are disjoint neighbourhoods U_x of x and U_y of y in X.

So let x, y be any two distinct points in S. Since $x, y \in X$ and X is Hausdorff, they have disjoint neighbourhoods $U_x \subseteq X$ and $U_y \subseteq X$, respectively. Then consider the sets $U_x \cap S$ and $U_y \cap S$. These are both open subsets of S, by definition of the subspace topology. The point x is contained in both U_x and S, so $x \in U_x \cap S$, and similarly $y \in U_y \cap S$. Finally, since the sets U_x and U_y are disjoint, their intersections with S must also be disjoint. Thus $U_x \cap S$ and $U_y \cap S$ are the desired disjoint neighbourhoods of x and y in S, and we conclude that S is Hausdorff.

2. (4 points) Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous **injective** map. Show that, if Y is Hausdorff, then X is Hausdorff.

Solution: Let x_1 and x_2 be any two distinct points of X. To prove that X is Hausdorff, we seek disjoint open neighbourhoods U_1 and U_2 of x_1 and x_2, respectively.

Since f is injective, $f(x_1) \neq f(x_2)$. Then, since Y is Hausdorff, there are disjoint open neighbourhoods V_1 of $f(x_1)$ and V_2 of $f(x_2)$.

Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. We will show that these are the desired neighbourhoods.

Since V_1 is open and f is continuous, $f^{-1}(V_1)$ is open by the definition of continuity. Since $f(x_1) \in V_1$, it follows that $x_1 \in f^{-1}(V_1)$ by definition of pre-image. Hence $f^{-1}(V_1)$ is an open neighbourhood of x_1. By the same reasoning, $f^{-1}(V_2)$ is an open neighbourhood of x_2. It remains to show that $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint.

Suppose that $x \in f^{-1}(V_1) \cap f^{-1}(V_2)$. But then by definition of pre-image, $f(x) \in V_1$ and $f(x) \in V_2$. This contradicts the assumption that V_1 and V_2 are disjoint. It follows that $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$. This concludes the proof that X is Hausdorff.