Name: ______________________ Score (Out of 9 points):

1. (6 points) Which of the following functions are continuous? Briefly explain your answer.

(a)
$$f : (\mathbb{R}, \text{Euclidean}) \to (\mathbb{R}, \text{cofinite})$$
$$f(x) = x$$

Solution: This function is continuous. Since f is the identity function on $\mathbb{R}$, $f^{-1}(U) = U$ for any subset $U \in \mathbb{R}$. So consider an open subset U of $(\mathbb{R}, \text{cofinite})$. If $U = \varnothing$, then $f^{-1}(U) = U$ is also an open subset of $(\mathbb{R}, \text{Euclidean})$. If $U \neq \varnothing$, then $f^{-1}(U) = U$ must be the complement of a finite set. But then U is also open in $(\mathbb{R}, \text{Euclidean})$. Thus $f^{-1}(U)$ is open for all open subsets U of $(\mathbb{R}, \text{cofinite})$, and we conclude that f is continuous.

(b)
$$f : (\mathbb{R}, \text{cofinite}) \to (\mathbb{R}, \text{cofinite})$$
$$f(x) = \sin(x)$$

Solution: This function is **not** continuous. A proper subset C of $(\mathbb{R}, \text{cofinite})$ is closed if and only if it is finite. So consider the closed subset $\{0\} \subseteq \mathbb{R}$. Its preimage under $\sin(x)$ is the infinite set $\{\ldots - 3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\}$. Thus by Worksheet #10 Problem 4(a), the function $\sin(x)$ is not continuous.

(c) Let $X = \{a, b\}$ with the topology $\mathcal{T} = \Big\{\varnothing, \{a\}, \{a, b\}\Big\}$.

$$f : X \to X$$
$$f(a) = b$$
$$f(b) = a$$

Solution: This function is **not** continuous. The subset $\{a\} \subseteq X$ is open, but its preimage $f^{-1}(\{a\}) = \{b\}$ is not open.

2. (3 points) Our concept of an accumulation point also makes sense in abstract topological spaces:

   Definition. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$. A point $x \in X$ is an *accumulation point* of A if and only if every neighbourhood of x contains a point of A distinct from x.

   Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$. Let $\mathcal{B}$ be a basis for $\mathcal{T}$. Prove that x is an accumulation point of A if and only if every basis element $B \in \mathcal{B}$ containing x contains a point of A distinct from x.

   Solution: Suppose first that every basis element $B \in \mathcal{B}$ containing x contains a point of A distinct from x. We will show that x is an accumulation pointt.

   Let U be a neighbourhood of x. By definition of a basis, U must be a union of basis elements. Thus there must be some basis element B with $x \in B \subseteq U$. By assumption, B contains a point $a \in A$ distinct from x. But then $a \in U$. Since U was an arbitrary open neighbourhood of x, this implies that x is an accumulation point of A as claimed.

   Now suppose that x is an accumulation point of A, and let $B \in \mathcal{B}$ be a basis element containing x. But then B is (by definition of a basis) an open neighbourhood of x, so B must contain an element $a \in A$ distinct from x, as claimed.