Name: ______________________ Score (Out of 6 points):

1. (3 points) For each of the following statements: if the statement is always true, write "True". Otherwise, state a counterexample. **No further justification needed.**

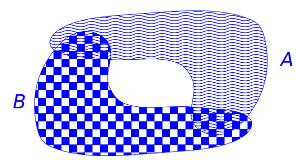
   Note: If the statement is not always true, you can receive partial credit for writing "False" without a counterexample.

   (a) Let A and B be nonempty subsets of a topological space $(X, \mathcal{T})$. If A and B are connected and $A \cap B$ is nonempty, then $A \cup B$ is connected.

   True. See Worksheet #14 Problem 6(b).

   (b) Let A and B be nonempty subsets of a topological space $(X, \mathcal{T})$. If A and B are connected and $A \cap B$ is nonempty, then $A \cap B$ is connected.

   False. For example, consider the following two subsets of $\mathbb{R}^2$ (with the Euclidean metric).



   (c) Let A and B be nonempty subsets of a topological space. If $A \cap B = \varnothing$, then $A \cup B$ is disconnected.

   False. For example, consider $A = (-\infty, 0)$ and $B = [0, \infty)$ as subsets of $\mathbb{R}$ with the Euclidean topology. Then A and B are disjoint but $A \cup B = \mathbb{R}$ is connected.

2. (3 points) Let X be a topological space. Suppose that A is a nonempty proper subset of X such that $\partial A = \varnothing$. Show that X is disconnected.

Solution. From Worksheet #14 Problem 3, to show that X is disconnected, it is enough to show that the nonempty proper subset A is both open and closed.

We know that $\operatorname{Int}(A) \subseteq A \subseteq \overline{A}$. By definition, $\partial A = \overline{A} \setminus \operatorname{Int}(A)$. So the assumption that $\partial A = \varnothing$ implies that $\operatorname{Int}(A) = \overline{A}$. But the only way this is possible is if each containment is equality:

$$\operatorname{Int}(A) = A = \overline{A}.$$

By Worksheet #13 Problem 1, the equality $\operatorname{Int}(A) = A$ implies that A is open, and by Worksheet #13 Problem 2, the equality $A = \overline{A}$ implies that A is closed.

Thus A is a nonempty proper subset of X that is both closed and open, and we conclude that X is disconnected.