1 Interior, closure, and boundary

Recall the definitions of interior and closure from Homework #7.

Definition 1.1. (Interior of a set in a topological space). Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *interior* of A to be the set

$$\text{Int}(A) = \{ \, a \in A \mid \text{there is some neighbourhood } U \text{ of } a \text{ such that } U \subseteq A \, \}.$$

You proved the following:

Proposition 1.2. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.*

- $\text{Int}(A)$ *is an open subset of X contained in A.*
- $\text{Int}(A)$ *is the largest open subset of A, in the following sense: If $U \subseteq A$ is open, then $U \subseteq \text{Int}(A)$.*

Definition 1.3. (Closure of a set in a topological space). Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *closure* of A to be the set

$$\overline{A} = \{ \, x \in X \mid \text{any neighbourhood } U \text{ of } x \text{ contains a point of } A \, \}.$$

You proved the following:

Proposition 1.4. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.*

- $\overline{A}$ *is a closed subset containing A.*
- $\overline{A}$ *is the smallest closed subset containing A, in the following sense: If C is a closed subset with $A \subseteq C$, then $\overline{A} \subseteq C$.*

We can similarly define the boundary of a set A, just as we did with metric spaces.

Definition 1.5. (Boundary of a set A). Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then the *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \text{Int}(A)$.

Example 1.6. Find the interior, closure, and boundary of the following subsets A of the topological spaces $(X, \mathcal{T})$.

(a) $X = \{a, b, c\}$, $\mathcal{T} = \Big\{ \varnothing, \{c\}, \{b, c\}, \{a, b, c\} \Big\}$, $A = \{a, c\}$.

(b) $X = \mathbb{R}$ with the cofinite topology, $A = (0, 1)$.

In-class Exercises

1. Let X be a topological space, and $A \subset X$. Prove the following.
   (a) A is open if and only if $A = \text{Int}(A)$.
   (b) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.
   (c) $\text{Int}(A) = \bigcup_{U \subseteq A,\, U \text{ open}} U.$

2. Let X be a topological space, and $A \subseteq X$. Prove the following.
   (a) A is closed if and only if $A = \overline{A}$.
   (b) $\overline{\overline{A}} = \overline{A}$.
   (c) $\overline{A} = \bigcap_{C \text{ closed},\, A \subseteq C} C.$

3. Let X be a topological space, and $A \subseteq X$.
   (a) Prove that $\partial A = \overline{A} \cap (\overline{X \setminus A})$.
   (b) Use this result to conclude that (i) ∂A is closed, and (ii) $\partial A = \partial(X \setminus A)$.
   (c) Prove the following.

   Theorem (An equivalent definition of ∂A). Let X be a topological space, and let $A \subseteq X$. Then

   $$\partial A = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{every open neighbourhood } U \text{ of } x \text{ contains at least one point of } A, \\ \text{and at least one point of } X \setminus A. \end{array} \right\}$$

   (d) Prove that every point of X falls into one of the following three categories of points, and that the three categories are mutually exclusive:
      (i) interior points of A;
      (ii) interior points of $X \setminus A$;
      (iii) points in the (common) boundary of A and $X \setminus A$.

4. **(Optional).** Let A be a subset of a topological space X. Prove the following.
   (a) $X \setminus \overline{A} = \text{Int}(X \setminus A)$.
   (b) $X \setminus \text{Int}(A) = \overline{X \setminus A}$.

5. **(Optional).** Suppose (X, d) is a metric space, and $A \subseteq X$. You proved on Quiz #4 that, if $x \in \overline{A}$, then there is some sequence of points $(a_n)_{n \in \mathbb{N}}$ in A that converge to x. In this problem, we will see that this property does **not** hold for general topological spaces.
   (a) Recall that the *co-countable* topology on $\mathbb{R}$ is

   $$\mathcal{T}_{cc} = \{\varnothing\} \cup \{\, U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is countable } \}.$$

   Let $A \subseteq \mathbb{R}$. What is $\overline{A}$ if A is (i) countable, or (ii) uncountable?
   (b) Let $A = (0, 1)$, so $\overline{A} = \mathbb{R}$. Show that, for any $x \in \overline{A} \setminus A$, there is **no** sequence of points in A that converge to x.
   (c) **Definition (First countable spaces).** A topological space $(X, \mathcal{T})$ is called *first countable* if each point $x \in X$ has a *countable neighbourhood basis.* This means, for each $x \in X$, there is a countable collection $\{N_i\}_{i \in \mathbb{N}}$ of neighbourhoods of x with the property that, if N is any neighbourhood of x, then there is some i such that $N_i \subseteq N$.

   Let X be a first countable space, and let $A \subseteq X$. Show that, given any $x \in \overline{A}$, there is some sequence of points in A that converges to x.