1 Compact topological spaces

Recall the definition of an open cover:

Definition 1.1. (Open covers; open subcovers.) Let $(X, \mathcal{T})$ be a topological space. A collection $\{U_i\}_{i \in I}$ of open subsets of X is an *open cover* of X if $X = \bigcup_{i \in I} U_i$. In other words, every point in X lies in some set U_i.

A sub-collection $\{U_i\}_{i \in I_0}$ (where $I_0 \subseteq I$) is an *open subcover* (or simply *subcover*) if $X = \bigcup_{i \in I_0} U_i$., In other words, every point in X lies in some set U_i in the subcover.

Definition 1.2. (Compact spaces; compact subspaces.) We say that a topological space $(X, \mathcal{T})$ is *compact* if **every** open cover of X has a finite subcover. A subset $A \subseteq X$ is called *compact* if it is compact with respect to the subspace topology.

Example 1.3. Let $(X, \mathcal{T})$ be a finite topological space. Then X is compact.

Example 1.4. Let X be a topological space with the indiscrete topology. Then X is compact.

Example 1.5. Let X be an infinite topological space with the discrete topology. Then X is **not** compact.

In-class Exercises

1. (a) Let X be a set with the cofinite topology. Prove that X is compact.
   (b) Let $X = (0, 1)$ with the topology induced by the Euclidean metric. Show that X is not compact.

2. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, and $f : X \to Y$ is a continuous map. Show that, if X is compact, then $f(X)$ is a compact subspace of Y. In other words, the continuous image of a compact set is compact.

3. Prove that a closed subset of a compact space is compact.

4. (a) Let X be a topological space with topology induced by a metric d. Prove that any compact subset A of X is bounded.
   (b) Suppose that $(X, \mathcal{T})$ is a **Hausdorff** topological space. Prove that any compact subset A of X is closed in X.
   (c) Consider $\mathbb{Q}$ with the Euclidean metric. Show that the subset $(-\pi, \pi) \cap \mathbb{Q}$ of $\mathbb{Q}$ is closed and bounded, but not compact.

5. **(Optional).** Determine which of the following topologies on $\mathbb{R}$ are compact.
   - Any topology $\mathcal{T}$ consisting of only finitely many sets.
   - the discrete topology
   - $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$

6. **(Optional).** Consider $\mathbb{R}$ with the topology $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$. Give necessary and sufficient conditions for a subset $C \subseteq \mathbb{R}$ to be compact.

7. **(Optional).** Let X be a nonempty set, and let x_0 be a distinguished element of X. Let
$$\mathcal{T} = \{A \subseteq X \mid x_0 \notin A \text{ or } X \setminus A \text{ is finite }\}.$$
   (a) Show that $\mathcal{T}$ defines a topology on X.
   (b) Verify that $(X, \mathcal{T})$ is Hausdorff.
   (c) Verify that $(X, \mathcal{T})$ is compact.

   This exercise shows that **any** nonempty set X admits a topology making it a compact Hausdorff topological space.

8. **(Optional)**. Let $K_1 \supseteq K_2 \supseteq \cdots$ be a descending chain of nonempty, closed, compact sets. Then $\bigcap_{n \in \mathbb{N}} K_n \neq \varnothing$.

9. **(Optional)**. Let X be a topological space, and let $A, B \subseteq X$ be compact subsets.
   (a) Suppose that X is Hausdorff. Show that $A \cap B$ is compact.
   (b) Show by example that, if X is not Hausdorff, $A \cap B$ need not be compact.
   Hint: Consider $\mathbb{R}$ with the topology $\{U \mid U \subseteq \mathbb{R},\ 0, 1 \notin U\} \cup \{\mathbb{R}\}$.

10. **(Optional).** Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, and $f : X \to Y$ is a closed map (this means that $f(C)$ is closed for every closed subset $C \subseteq X$). Suppose that Y is compact, and moreover that $f^{-1}(y)$ is compact for every $y \in Y$. Prove that X is compact.