1 Homeomorphisms

Our definition of homeomorphism (Homework #3 Problem 2) generalizes to abstract topological spaces:

Definition 1.1. (Homeomorphisms of topological spaces). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is a *homeomorphism* if

- f is continuous,
- f has an inverse $f^{-1} : Y \to X$, and
- f^{-1} is continuous.

The topological space $(X, \mathcal{T}_X)$ is said to be *homeomorphic* to the topological space $(Y, \mathcal{T}_Y)$ if there exists a homeomorphism $f : X \to Y$.

Two topological spaces are considered "the same" topological space if and only if they are homeomorphic.

In-class Exercises

1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Explain the sense in which an homeomorphism $f : X \to Y$ defines a bijection between the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$.

2. (a) Let $(X, \mathcal{T}_X)$ be a topological space. Show that X is homeomorphic to itself.

   (b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a homeomorphism. Explain why $f^{-1} : Y \to X$ is also a homeomorphism. Conclude that X is homeomorphic to Y if and only if Y is homeomorphic to X. (We simply call the spaces "homeomorphic topological spaces").

   (c) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Show that, if X is homeomorphic to Y, and Y is homeomorphic to Z, then X is homeomorphic to Z.

   This exercise shows that homeomorphism defines an *equivalence relation* on topological spaces.

3. Determine which of the following properties are preserved by homeomorphism. In other words, suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic topological spaces. For each of the following properties P, prove or give a counterexample to the statement "X has property P if and only if Y has property P".
   (For some properties to be defined, you will need to assume that X and Y are metric spaces.)

   (i) discrete topology
   (ii) indiscrete topology
   (iii) T_1
   (iv) Hausdorff
   (v) regular
   (vi) number of connected components
   (vii) path-connected
   (viii) complete
   (ix) sequentially compact
   (x) compact
   (xi) bounded
   (xii) metrizable

Properties that are preserved by homeomorphisms are called *homeomorphism invariants*, *topological invariants*, or *topological properties* of a topological space.

4. Use the results of Problem 3 to explain why the following pairs of spaces are *not* homeomorphic.
   (a) $(0, 1)$ and $[0, 1]$ (with the Euclidean metric)
   (b) $\mathbb{R}$ with the Euclidean metric and $\mathbb{R}$ with the cofinite topology
   (c) $(0, 2)$ and $(0, 1] \cup (2, 3)$ (with the Euclidean metric)

5. **(Optional).**
   (a) Prove that a map $f : X \to Y$ of topological spaces is a homeomorphism if and only if it is continuous, invertible, and open.
   (b) Prove that a map $f : X \to Y$ of topological spaces is a homeomorphism if and only if it is continuous, invertible, and closed.

6. **(Optional).** Let $f : X \to Y$ be a homeomorphism, and let $A \subseteq X$. Prove that f restricts to a homeomorphism $f|_A : A \to f(A)$ between the subspaces A and $f(A)$.

7. **(Optional).**
   (a) Prove that two spaces X and Y with the discrete topology are homeomorphic if and only if they have the same cardinality.
   (b) Prove that two spaces X and Y with the cofinite topology are homeomorphic if and only if they have the same cardinality.

8. **(Optional).** Let $X \times Y$ be the product of a space X and a nonempty space Y, endowed with the product topology. Fix $y_0 \in Y$. Prove that X is homeomorphic to the subspace $X \times \{y_0\} \subseteq X \times Y$.

9. **(Optional).** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $F : X \to Y$ a continuous function. Recall that the *graph* G of F is the set

$$G = \{(x, f(x)) \mid x \in X\}$$

viewed as a subspace of $X \times Y$ with the product topology. Prove that G is homeomorphic to X.