## 1 Open and closed sets

**Definition 1.1.** (Open ball of radius r about  $x_0$ .) Let (X, d) be a metric space, and  $x_0 \in X$ . Let  $r \in \mathbb{R}$ , r > 0. We define the *open ball of radius* r *about*  $x_0$  as the subset of X

$$B_r(x_0) = \{ x \in X \mid d(x_0, x) < r \} \subseteq X.$$

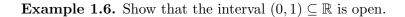
**Example 1.2.** Let  $X = \mathbb{R}$  with the usual Euclidean metric d(x,y) = |x-y|. What is  $B_1(0)$ ? What is  $B_2(-6)$ ?

**Example 1.3.** Let  $X = \mathbb{R}^2$  with the usual Euclidean metric. Draw  $B_2(1,0)$ .

**Definition 1.4.** (Interior points; open sets in a metric space.) Let (X, d) be a metric space, and let  $U \subseteq X$  be a subset of X. A point  $x \in U$  is called an *interior point of* U if there is some radius  $r_x \in \mathbb{R}$ ,  $r_x > 0$ , so that  $B_{r_x}(x) \subseteq U$ .

The set  $U \subseteq X$  is called *open* if every point  $x \in U$  is an interior point of U.

**Example 1.5.** Show that the interval  $[0,1) \subseteq \mathbb{R}$  is not open.



**Proposition 1.7.** Let (X,d) be a metric space,  $x_0 \in X$  and  $0 < r \in \mathbb{R}$ . Then the ball  $B_r(x_0)$  is an open subset of X.

Proof.

**Definition 1.8.** (Closed sets in a metric space.) A subset  $C \subseteq X$  is *closed* if its complement  $X \setminus C$  is open.

## In-class Exercises

- 1. Let (X, d) be a metric space. Solve (with justification) the following:
  - (a) Is X open?
- (b) Is  $\varnothing$  open?
- (c) Is X closed?
- (d) Is  $\varnothing$  closed?
- 2. Consider  $\mathbb{R}$  with the Euclidean metric. Find an example of a subset of  $\mathbb{R}$  that is ...
  - (a) open and not closed,

(c) both open and closed,

(b) closed and not open,

- (d) neither open nor closed.
- 3. Let  $X = \mathbb{R}^2$ . Sketch the balls  $B_1(0,0)$  and  $B_2(0,0)$  for each of the following metrics on  $\mathbb{R}^2$ . Denote  $\overline{x} = (x_1, x_2)$  and  $\overline{y} = (y_1, y_2)$ .
  - (a)  $d(\overline{x}, \overline{y}) = ||\overline{x} \overline{y}|| = \sqrt{(x_1 y_1)^2 + (x_2 y_2)^2}$
  - (b)  $d(\overline{x}, \overline{y}) = |x_1 y_1| + |x_2 y_2|$
  - (c)  $d(\overline{x}, \overline{y}) = \max\{|x_1 y_1|, |x_2 y_2|\}$
  - (d)  $d(\overline{x}, \overline{y}) = \begin{cases} 0, & \overline{x} = \overline{y} \\ 1, & \overline{x} \neq \overline{y} \end{cases}$
- 4. Let  $\{U_i\}_{i\in I}$  denote a collection of open sets in a metric space (X,d).
  - (a) Prove that the union  $\bigcup_{i \in I} U_i$  is an open set. Do not assume that I is necessarily finite, or countable!
  - (b) Show by example that the intersection  $\bigcap_{i\in I} U_i$  may not be open. (This means, give an example of a metric space (X,d) and a collection of open sets  $U_i\subseteq X$ , and prove that  $\bigcap_{i\in I} U_i$  is not open).
  - (c) Now assume we have a **finite** collection  $\{U_i\}_{i=1}^n$  of open sets in a metric space. Prove that the intersection  $\bigcap_{i=1}^n U_i$  is open.
- 5. (Optional).
  - (a) Rigorously verify that the sets  $\{1\}$ ,  $[1, \infty)$ , and  $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\} \cup \{0\}$  are all closed subsets of  $\mathbb{R}$  (with the Euclidean metric).
  - (b) Consider  $\mathbb{R}$  with the Euclidean metric. Is the subset  $\mathbb{Q} \subseteq \mathbb{R}$  open? Is it closed?
  - (c) Recall that C(0,2) is the set of continuous functions from the closed interval [0,2] to  $\mathbb{R}$ , and that

$$d_{\infty}: \mathcal{C}(0,2) \times \mathcal{C}(0,2) \longrightarrow \mathbb{R}$$
$$d(f,g) = \sup_{x \in [0,2]} |f(x) - g(x)|$$

defines a metric on C(0,2). Determine whether the subset  $\{f(x) \in C(0,2) \mid f(1) = 0\}$  is closed, open, neither, or both.

- 6. (Optional). Let X be a finite set, and let d be any metric on X. What can you say about which subsets of X are open? Which subsets of X are closed?
- 7. (Optional). Let (X, d) be a metric space, and let  $x \in X$ . Prove that the singleton set  $\{x\}$  is closed.