

## 1 Continuous functions on metric spaces

**Definition 1.1. (Continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .)** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Then  $f$  is *continuous at a point*  $x \in \mathbb{R}$  if ...

The function  $f$  is called *continuous* if it is continuous at every point  $x \in \mathbb{R}$ .

Rephrased:

How can we generalize this definition to general metric spaces?

**Definition 1.2. (Continuous functions on metric spaces.)** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f : X \rightarrow Y$  be a function. Then  $f$  is *continuous at a point*  $x \in X$  if ...

The function  $f$  is called *continuous* if it is continuous at every point  $x \in X$ .

Rephrased:

## In-class Exercises

1. In this question, we will prove the following result:

**Theorem (Continuous functions.)** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and let  $f : X \rightarrow Y$  be a function. Then  $f$  is continuous if and only if, given any open set  $U \subseteq Y$ , its preimage  $f^{-1}(U) \subseteq X$  is open.

- (a) Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and let  $f : X \rightarrow Y$  be a continuous function. Suppose that  $U \subseteq Y$  is an open set. Prove that  $f^{-1}(U)$  is open.
- (b) Suppose that  $f$  is a function with the property that, for every open set  $U \subseteq Y$ , the preimage  $f^{-1}(U)$  is an open set in  $X$ . Show that  $f$  is continuous.
2. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous functions between metric spaces. Show that the composite

$$g \circ f : X \rightarrow Z$$

is continuous. *Hint:* With our new criterion for continuity, this argument can be quite quick!

3. **(Optional)** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. A function  $f : X \rightarrow Y$  is called an *isometric embedding* if it is “distance-preserving” in the sense that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

- (a) Give an intuitive description, with pictures, of what it means for a map to be an isometric embedding.
- (b) Show that an isometric embedding is continuous.
- (c) Show that an isometric embedding is always injective.
- (d) Show that map

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (x, 0) \end{aligned}$$

is an isometric embedding of  $\mathbb{R}$  into  $\mathbb{R}^2$  (each with the Euclidean metric).

- (e) Consider the linear function  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  given by the map  $f(x) = (x, mx + b)$  for  $m, b \in \mathbb{R}$ , so  $f$  maps the real line to the graph of the function  $mx + b$ . For which values of  $m$  and  $b$  is this an isometric embedding of Euclidean spaces?
- (f) For functions  $f(x) = (x, mx + b)$  that are not isometric embeddings, can you find a different parameterization of this line that is an isometric embedding? In other words, can you find an isometric embedding  $g : \mathbb{R} \rightarrow \mathbb{R}^2$  whose image is the set  $\{(x, mx + b) \mid x \in \mathbb{R}\}$ ?
- (g) Let  $X = \{a, b, c\}$  be a 3-point set. Find examples of metrics on  $X$  so that the resulting metric space can and cannot be isometrically embedded in Euclidean space  $\mathbb{R}^2$ . Can you find necessary and sufficient conditions on the metric on  $X$  to guarantee the existence of an isometric embedding of  $X$  into  $\mathbb{R}^2$ ?
- (h) Show that the image of any isometric embedding from  $\mathbb{R}$  into  $\mathbb{R}^2$  must be a straight line.