1 Sequential Compactness

Definition 1.1. (Subsequences.) Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. Let

$$0 < n_1 < n_2 < \cdots < n_i < \cdots$$

be an infinite sequence of strictly increasing natural numbers. Then $(a_{n_i})_{i\in\mathbb{N}}$ is called a *subsequence* of $(a_n)_{n\in\mathbb{N}}$.

Proposition 1.2. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X, d) that converges to a point a_∞. Show that any subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a_∞.*

Definition 1.3. (Sequantially compact metric spaces; sequentially compact subsets.) A metric space (X, d) is called *sequentially compact* if every sequence in X has a convergent subsequence. Similarly, a subset $S \subseteq X$ is *sequentially compact* if every sequence of points in S has a subsequence that converges to a point in S.

By convention, the empty set $\varnothing$ is considered sequentially compact.

Example 1.4. Give examples of sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers with the following properties:

(a) The sequence $(a_n)_{n\in\mathbb{N}}$ has a subsequence that converges to 0, and a subsequence that converges to 1.

(b) The sequence $(a_n)_{n\in\mathbb{N}}$ has no convergent subsequence.

(c) The sequence $(a_n)_{n\in\mathbb{N}}$ is an unbounded sequence with a subsequence that converges to 0.

In-class Exercises

1. Suppose that (X, d) is a sequentially compact metric space, and that $C \subseteq X$ is a closed subset. Prove that C is sequentially compact.

2. Let (X, d) be a metric space.
   (a) Prove that every sequentially compact subset of X is closed.
   (b) **Definition (Bounded subset.)** Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.

   Prove that every sequentially compact subset of X is bounded.
   (c) Is it true that every closed and bounded subset of a metric space is sequentially compact?

3. **(Optional).** Let (X, d) be a metric space, and $A \subseteq X$ a subset. Show that A is bounded if and only if there is some $M \in \mathbb{R}$ so that $d(a, b) \leq M$ for all $a, b \in A$.

4. **(Optional).** Let X be a metric space with the discrete metric. Give necessary and sufficient conditions for a subset of X to be sequentially compact.

5. **(Optional).** Recall that the space $\mathcal{C}([a, b])$ of continuous functions $f : [a, b] \to \mathbb{R}$ is a metric space with metric
$$d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$
   (a) Show that $\mathcal{C}([a, b])$ is unbounded. Conclude that it is **not** sequentially compact.
   (b) Consider the subset
$$C = \{f(x) \in \mathcal{C}([0, 1]) \mid |f(x)| \leq 1 \text{ for all } x \in [0, 1]\}.$$
   Is C sequentially compact?

6. **(Optional).** Let X be a metric space, and let $a_\infty \in X$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X with the property that every subsequence of $(a_n)_{n \in \mathbb{N}}$ has a subsequence that converges to a_∞. Prove that $(a_n)_{n \in \mathbb{N}}$ converges to a_∞.

7. **(Optional).**
   (a) Consider the sequence of real numbers
$$0,\ 1,\ 0,\ \frac{1}{2},\ 1,\ 0,\ \frac{1}{3},\ \frac{2}{3},\ 1,\ 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1,\ 0,\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ 1,\ \ldots$$
   What real numbers can be realized as the limit of a subsequence of this sequence?
   (b) Is it possible to construct a sequence $(a_n)_{n \in \mathbb{N}}$ with subsequences converging to every real number $r \in \mathbb{R}$?