1 Product Metrics

Definition 1.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Then their Cartesian product $X \times Y$ has a metric space structure, defined by the metric

$$\begin{aligned} d_{X\times Y} :\ (X \times Y) \times (X \times Y) &\longrightarrow \mathbb{R} \\ d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) &= \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}. \end{aligned}$$

We will call $d_{X\times Y}$ the *product metric* on $X \times Y$.[1]

Example 1.2. Consider $\mathbb{R}$ with the Euclidean metric. What is the product metric on $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$?

In-class Exercises

1. Verify that $d_{X\times Y}$ does in fact define a metric on $X \times Y$.
2. (a) Prove that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then $U \times V$ is an open subset of $X \times Y$.
   (b) Let $U \subseteq X \times Y$ be an open set, and let $(x, y) \in U$. Show that there is a neighbourhood U_x of x in X and a neighbourhood U_y of y in Y so that $U_x \times U_y \subseteq U$.
3. **Definition (Projection maps).** For a product of sets $X \times Y$, the maps

$$\begin{aligned} \pi_X : X \times Y &\to X \\ \pi_X(x, y) &= x \end{aligned} \qquad\qquad \begin{aligned} \pi_Y : X \times Y &\to Y \\ \pi_Y(x, y) &= y \end{aligned}$$

   are called the *projection onto* X and the *projection onto* Y, respectively.

   (a) Let (X, d_X) and (Y, d_Y) be topological spaces, and endow their product $X \times Y$ with the product metric $d_{X\times Y}$. Show that the projection map

$$\pi_X : (X \times Y, d_{X\times Y}) \longrightarrow (X, d_X)$$

   is continuous. (The same argument, which you do not need to repeat, shows that the map π_Y is continuous).

   (b) Prove that the projection map π_X is open. (The same argument shows π_Y is open).

   Hint: Use Question 2.
4. **(Optional).** A map is called *closed* if the image of every closed set is closed. Prove or find a counterexample: the projection map $\pi_X : X \times Y \to X$ is always a closed map.

[1] *Remark:* In light of Problem 5(c), any of the metrics in Problem 5 – among others – are sometimes called the *product metrics* on $X \times Y$. For clarity in this course we will use this term specifically for the metric $d_{X\times Y}$.

5. **(Optional).** Let (X, d_X) and (Y, d_Y) be metric spaces.

   (a) Show that each of the following functions also defines a metric on $X \times Y$.

$$d_1 : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

$$d_\infty : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max \{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

   (b) Show that these metrics on $X \times Y$ are all *topologically equivalent* to the metric $d_{X \times Y}$. This means that a subset of $X \times Y$ open with respect to one metric if and only if it is open with respect to the other.

6. **(Optional)**. Let (X_i, d_i) be metric spaces for $i \in \mathbb{N}$. Can you find a natural way to define a metric on the infinite product $\prod_{i \in \mathbb{N}} X_i = X_1 \times X_2 \times X_3 \times \cdots$?
   There are multiple ways to do this, which are not topologically equivalent!

7. **(Optional).** Let (X, d) be a metric space. Show that the map $d : X \times X \to \mathbb{R}$ is continuous with respect to the product metric on $X \times X$ and the standard topology on $\mathbb{R}$.