1 Complete metric spaces

Definition 1.1. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

Example 1.2. Give an example of a metric space that is not complete.

In-class Exercises

1. Suppose that a metric space (X, d) is sequentially compact. Show that (X, d) is complete.

2. Recall the following result from real analysis (which you do not need to prove):

   Theorem (Bolzano–Weierstrass). Let $S \subseteq \mathbb{R}^n$ be a bounded infinite set. Then S has an accumulation point $x \in \mathbb{R}^n$.

   Prove the following:

   Theorem (Sequential compactness in $\mathbb{R}^n$). Consider the space $\mathbb{R}^n$ with the Euclidean metric. Let $S \subseteq \mathbb{R}^n$ be a (finite or infinite) subset. Then S is sequentially compact if and only if S is closed and bounded.

3. Prove the following theorem.

   Theorem ($\mathbb{R}^n$ is complete). The space $\mathbb{R}^n$ is complete with respect to the Euclidean metric.

   Hint: Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$ (with the Euclidean metric). First prove that, for some $R > 0$, the set $\{a_n \mid n \in \mathbb{N}\}$ is contained in the set $\{x \in \mathbb{R}^n \mid ||x|| \leq R\}$. Then use Problems 1 and 2.

4. **(Optional)** Let X be a nonempty set with the discrete metric. Under what conditions is X complete?

5. **(Optional)** Let X and Y be metric spaces. Suppose that X is complete.
   (a) Let $f : X \to Y$ be a continuous function. Must its image $f(X)$ be complete?
   (b) Suppose $f : X \to Y$ is a homeomorphism. Must Y be complete?
   (c) Suppose that $f : X \to Y$ is an isometric embedding. Must $f(X)$ be complete?

6. **(Optional)** A subset A of a metric space X is *dense* if $\overline{A} = X$. An *isometry* is a bijective isometric embedding.

   Definition (Completion). Let X be a metric space. The *completion* of X is a complete metric space Y along with an isometric embedding $h : X \to Y$ such that $h(X)$ is dense in Y.

   In this question, we will construct the completion of X, and verify that it is unique up to isometry.

   (a) Let A be a dense subset of metric space Z. Show that, if every Cauchy subsequence in A converges in Z, then Z is complete.

   (b) Let (X, d) be a metric space. Let $\tilde{X}$ denote the set of Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ in X. Let Y denote the equivalences classes defined by the equivalence relation on $\tilde{X}$,

   $$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \qquad \iff \qquad d(x_n, y_n) \xrightarrow{n\to\infty} 0.$$

   Verify that this is indeed an equivalence relation.

   (c) For $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, let $[\mathbf{x}]$ and $[\mathbf{y}]$ denote the corresponding equivalence classes. Define

   $$D : Y \times Y \to \mathbb{R}$$
   $$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

   Show that D is well-defined, that is, its value does not depend on the choice of representative of the equivalence class.

   (d) Show that D defines a metric on Y.

   (e) Define

   $$h : X \to Y$$
   $$x \mapsto [(x)_{n\in\mathbb{N}}]$$

   sending a point x to the constant sequence $(x)_{n\in\mathbb{N}}$. Show that h is an isometric embedding.

   (f) Show that, for any Cauchy sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, the sequence $(h(x_n))_{n\in\mathbb{N}}$ converges in Y to $[\mathbf{x}]$. Conclude that $h(X)$ is dense ins Y.

   (g) Further conclude that every Cauchy sequence in $h(X)$ must converge in Y, and thus by part (a) the space (Y, D) is complete. This shows that Y is the completion of X.

   (h) Show that this completion is unique, in the sense of the following theorem.

   Theorem (The completion of X is unique up to isometry). Let $h : X \to Y$ and $h' : X \to Y'$ be isometric embeddings of the metric space (X, d) into complete metric spaces (Y, D) and (Y', D'), respectively, with dense image. Then there is an isometry of $(\overline{h(X)}, D)$ and $(\overline{h'(X)}, D')$ that equals $h' \circ h^{-1}$ when restricted to the subspace $h(X)$.