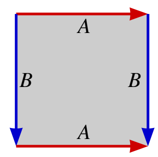
Recommended reading: Munkres Section 23–25.

Roughly similar content:
Notes by John Rognes, Section 3.1-3.3 https://www.uio.no/studier/emner/matnat/math/MAT4500/h13/topology.pdf

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. We can construct a torus T as a quotient of the unit square $[0,1]\times[0,1]$ by imposing the equivalence relation that $(0,y)\sim(1,y)$ for all y and $(x,0)\sim(x,1)$ for all x, as shown. Let $p:[0,1]\times[0,1]\to T$ be the quotient map.



   (a) Use the universal property of the quotient to explain why the map

$$\begin{aligned} g : [0,1]\times[0,1] &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \sin(2\pi x) \end{aligned}$$

   factors through the quotient map p (this means there is a map f with $f\circ p = g$), descending to a well-defined, continuous map $f : T\to\mathbb{R}$.

   (b) Decide which of the following maps $g:[0,1]\times[0,1]\to\mathbb{R}$ descend to a well-defined and continuous map $f:T\to\mathbb{R}$.

$$g(x,y)=\sin(2\pi x)\cos(2\pi y) \qquad g(x,y)=0 \qquad g(x,y)=y\sin(2\pi x)$$

$$g(x,y)=\left|x-\frac{1}{2}\right| \qquad g(x,y)=\begin{cases} 0, & (x,y)=(0,0),(1,0),(0,1),(1,1), \\ 1, & \text{otherwise.}\end{cases}$$

2. (a) Let $X\neq\varnothing$ have the indiscrete topology. Show that X is both connected and path-connected.
   (b) Let X be a set of at least two elements with the indiscrete topology. Show that X is neither connected nor path-connected.

3. Suppose x,y are points in a topological space X joined by a path $f:[a,b]\to X$. Show that there is a path of the form $g:[0,1]\to X$ from x to y. Conclude that it would be equivalent to define (as some sources do) path-connectedness in terms of paths of the form $[0,1]\to X$.

4. Let $X=\{a,b,c\}$. Determine which of the following topologies on X are connected, and which are path-connected.

   (i) $\{\varnothing,\{a,b,c\}\}$
   (ii) $\{\varnothing,\{c\},\{a,b,c\}\}$
   (iii) $\{\varnothing,\{c\},\{a,b\},\{a,b,c\}\}$
   (iv) $\{\varnothing,\{c\},\{b,c\},\{a,b,c\}\}$
   (v) $\{\varnothing,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$
   (vi) $\{\varnothing,\{b\},\{c\},\{a,b\},\{b,c\},\{a,b,c\}\}$

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. (a) **Definition (linear continuum).** Let L be a totally ordered set with at least two elements. Then L is called a *linear continuum* if:
   - L has the least upper bound property: every nonempty subset of L with an upper bound has a **least** upper bound;
   - if $x<y$ for $x,y\in L$, then there exists some $z\in L$ with $x<z<y$.

Let L be a linear continuum. Prove that L is connected, as are intervals and half-rays in L.
Remark: The most important example of a linear continuum is $\mathbb{R}$ (or subintervals of $\mathbb{R}$), and this is the reason we wish to study them. Approaching the following questions from the viewpoint of linear continua help us identify precisely which properties of $\mathbb{R}$ we are using.

(b) Let X be an ordered set with the order topology. Show that, if X is connected, then X is a linear continuum.

(c) Prove the following theorem.

> **Theorem (Intermediate Value Theorem).** Let $f : X \to Y$ be a continuous map from a connected space X to an ordered set Y with the order topology. Suppose $a, b \in X$ and $r \in Y$ is a point lying between $f(a)$ and $f(b)$. Prove that there is a point c such that $f(c) = r$.

Remark: Part (a) then implies the following special case of the Intermediate Value Theorem (familiar from real analysis): Let $[a, b] \subseteq \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. If r is any number between $f(a)$ and $f(b)$, then there is some $c \in [a, b]$ with $f(c) = r$.

(d) Prove that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. (In other words, show that there is some $x \in [0, 1]$ so that $f(x) = x$).
Hint: Consider the function $g : [0, 1] \to \mathbb{R}$ given by $g(x) = f(x) - x$.

2. Let X be a topological space.

(a) Let $f : [a, b] \to X$ be path from x to y and let $g : [c, d] \to X$ be a path from y to z. Show that the following function is a path from x to z.

$$h : [a, b + d - c] \longrightarrow X$$
$$h(x) = \begin{cases} f(x), & a \leq x \leq b \\ g(x - b + c), & b \leq x \leq b + d - c \end{cases}$$

Hint: Homework 6 # 2(b). You may use without proof the fact that linear functions $\mathbb{R} \to \mathbb{R}$ are continuous.

(b) Show that the relation

$$x \sim y \quad \iff \quad \text{there is a path from } x \text{ to } y$$

is an equivalence relation on X. Its equivalence classes are called the *path components* of X.

(c) In a sentence, explain why every connected component of X is a union of path components.

(d) Prove the following.

> **Theorem (Connected components vs. path components).** Let X be a topological space. If X is locally path connected, then the connected components and the path components of X are the same.

3. (a) Show that none of the spaces $[0, 1]$, $(0, 1]$, and $(0, 1)$ are homeomorphic. Similarly, show that $(0, 1)$ and the unit circle are not homeomorphic. *Hint:* What happens when you remove a point from these spaces?

(b) Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$.

4. We proved in class that a finite product of connected spaces is connected. In this problem we will investigate infinite products of connected spaces. Let $\mathbb{R}^\omega = \prod_{\mathbb{N}} \mathbb{R}$.

(a) Consider $\mathbb{R}^\omega$ with the box topology. Prove that the set B of all bounded sequences of real numbers, and the set A of all unbounded sequences of real numbers, separate $\mathbb{R}^\omega$. Conclude that $\mathbb{R}^\omega$ is disconnected in the box topology.

(b) Does this argument work to show that $\mathbb{R}^\omega$ is disconnected in the uniform topology?

(c) Consider $\mathbb{R}^\omega$ with the product topology. Show that $\mathbb{R}^\omega$ is path-connected, and therefore connected.

5. Determine whether the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected or path-connected.