Recommended reading: Munkres Section 26-28.

Roughly similar content: Hatcher, Chapter 3 https://pi.math.cornell.edu/ hatcher/Top/TopNotes.pdf.

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Show that, if a space X is locally path-connected at $x \in X$, then X is locally connected at x. Conclude that a locally path-connected space is locally connected.
2. Show that a space X is locally (path-)connected if and only if X has a basis of open sets that are (path-)connected.
3. Let X be a topological space, and $A \subseteq X$. Recall that, by 'a connected component of A', we mean a connected component of the space A in the subspace topology.
   (a) Show by example that two points a and b in A can be in different connected components of A, but the same connected component of X.
   (b) Suppose that a and b are in the same connected component of A. Can they be in different connected components of X?
4. Finish our proof in class of the following result:

   Let X be a topological space and $Y \subseteq X$ a subset. Then Y is compact in the subspace topology if and only if every collection $\mathcal{U} = \{U_i\}_{i \in I}$ of subsets of open subsets of X covering Y contains a finite subcover $U_{i_1}, \ldots U_{i_n}$ of Y.

5. Determine whether $\mathbb{R}$ is compact when given each of the following topolgies.
   (a) the discrete topology
   (b) the indiscrete topology
   (c) the standard topology
   (d) $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   (e) the cofinite topology
   (f) the cocountable topology
   (g) $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$
   (h) $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. In this question we will investigate the relationship between 'closed' and 'compact'.
   (a) Prove the following result.

   Theorem. Every closed subspace of a compact space is compact.

   (b) Give an example of a space X and a subspace $Y \subseteq X$ that is compact but not closed.
   (c) Let X be a Hausdorff space, and let $Y \subseteq X$ a compact subspace. Let $x_0 \in X \setminus Y$. Show that there are disjoint open subsets U and V of X such that $x_0 \in U$ and $Y \subseteq V$. Use this result to deduce the following.

   Theorem. Every compact subspace of a Hausdorff space is closed.

2. In this problem, we will show that a product of topological spaces X and Y is compact if and only if X and Y are compact.
   (a) Let X and Y be topological spaces, and that their Cartesian product $X \times Y$ is compact with respect to the product topology. Prove that X and Y are compact.

(b) Let X and Y be compact topological spaces. Let $\mathcal{U}$ be any open cover of $X \times Y$ (with the product topology). For this exercise, we will call a subset $A \subseteq X$ *good* if $A \times Y$ is covered by a finite subcollection of open sets in $\mathcal{U}$. Our goal is to show that X is good.

(i) Suppose that $A_1, \ldots, A_r$ is a finite collection of good subsets of X. Show that their union is good.

(ii) Fix $x \in X$. For each $y \in Y$, explain why it is possible to find open sets $U_y \in X$ and $V_y \in Y$ so that $(x, y) \in U_y \times V_y$ and $U_y \times V_y$ is contained in some open set in $\mathcal{U}$.

(iii) Explain why there is a finite list of points $y_1, \ldots, y_n \in Y$ so that the sets $\{V_{y_1}, \ldots, V_{y_n}\}$ cover Y.

(iv) Define

$$U_x = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}.$$

Show that U_x is a good set, and is an open subset of X containing x. This shows that every element $x \in X$ is contained in a good open set U_x.

(v) Consider the collection of open subsets $\{U_x \mid x \in X\}$ of X. Use the fact that X is compact to conclude that X is good.

3. (a) Prove the following theorem.

Theorem (Closed intervals are compact). Let X be a totally ordered set with the least upper bound property. Show that, in the order topology, for all $a, b \in X$, the closed interval $[a, b]$ is compact.

This shows in particular that closed intervals $[a, b]$ in $\mathbb{R}$ are compact.

You may wish to use the following steps. Let $\mathcal{U}$ be an open cover of $[a, b]$.

(i) Show that, if $x \in [a, b]$, $x \neq b$, then there exists a point $y > x$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{U}$.

(ii) Let $C = \{\ y \in (a, b] \mid [a, y]$ can be covered by finitely many elements of $\mathcal{U}\}$.
Then C is nonempty, and so has a least upper bound c.

(iii) The least upper bound c is contained in C.

(iv) The least upper bound $c = b$.

(b) Prove the following theorem.

Theorem (Compact subspaces of $\mathbb{R}^n$). Consider $\mathbb{R}^n$ with the Euclidean metric. A subspace $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Hint: First use Problems 2 (b) and 3 (a) to show that $[-N, N]^n \subseteq \mathbb{R}^n$ is compact.

(c) Prove the following theorem.

Theorem (Extreme value theorem). Let $f : X \to Y$ be a continuous function from a compact set X to a totally ordered set Y with the order topology. Then f achieves its maximum and minimum values. In other words, there exist points $c, d \in X$ so that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

4. **Definition (Diameter of a bounded set).** Let A be a bounded subset of a metric space (X, d). Then the *diameter* of A is defined to be

$$\sup\{\ d(a, b) \mid a, b \in A\}.$$

Definition (Lebesgue number of an open cover). Let $\mathcal{U}$ be an open cover of a metric space (X, d). Then a *Lebesgue number* for $\mathcal{U}$, if it exists, is a number $\delta > 0$ such that for every subset S of X of diameter less than δ, there is an element of $\mathcal{U}$ containing S.

Suppose that X is a sequentially compact metric space. Prove that every open cover $\mathcal{U}$ of X has a Lebesgue number $\delta > 0$.

5. **(Bonus).**

Definition (Real projective n-space). Projective space $P^n(\mathbb{R})$ is defined to be the quotient space of $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by the equivalence relation

$$\mathbf{x} \sim \alpha\mathbf{x} \qquad \text{for all } \alpha \in \mathbb{R} \setminus \{0\} \text{ and } \mathbf{x} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}.$$

(a) Show that $P^n(\mathbb{R})$ is *locally Euclidean*, in the sense that every point of $P^n(\mathbb{R})$ has a neighbourhood homeomorphic to $\mathbb{R}^n$.

(b) Show that $P^n(\mathbb{R})$ is compact.

(c) Let V be a vector space over a field $\mathbb{F}$, and define the corresponding projective space $P(V)$ to be the quotient of $V \setminus \{\mathbf{0}\}$ by the relation

$$\mathbf{x} \sim \alpha\mathbf{x} \qquad \text{for all } \alpha \in \mathbb{F} \setminus \{0\} \text{ and } \mathbf{x} \in V \setminus \{\mathbf{0}\}.$$

Show that any injective linear map $L : V \to W$ of finite dimensional vector spaces induces a map $P(V) \to P(W)$. When $\mathbb{F}$ is a topological field, show that the induced map is continuous with respect to the quotient topology on $P(V)$ and $P(W)$. In particular, show that $\mathrm{GL}_{n+1}(\mathbb{R})$ acts on $P^n(\mathbb{R})$ by continuous maps, and find the kernel of this action.