Recommended reading: Munkres Section 29-30.

Roughly similar content:
https://faculty.etsu.edu/gardnerr/5357/notes/Munkres-29.pdf
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Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X be a metric space. Verify that any ϵ-ball $B_\epsilon(x) \subseteq X$ has diameter 2ϵ.
2. Let X be a space.
   (a) Show that the finite union of compact subspaces of X is compact.
   (b) Show by example that a countable union of compact subspaces need not be compact.
3. Let X be a set with the discrete topology. Show that X is locally compact.
4. Let X be a locally compact space, and let $f : X \to Y$ be both continuous and open. Show that $f(X)$ is locally compact.
5. Prove the following. *Hint:* Problem #3 (f).
   (a) The one-point compactification of $\mathbb{R}$ is homeomorphic to the unit circle S^1.
   (b) The one-point compactification of $\mathbb{N}$ is homeomorphic to the subspace $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$.
6. (a) Let X be a topological space, and $\mathcal{B}$ a basis for its topology. Fix $x \in X$. Show that the set of neighbourhoods od x contained in $\mathcal{B}$ form a local basis at x.
   (b) Conclude that any second countable space is first countable.
7. (a) Show that a subspace of a first countable (respectively, second countable) space is also first countable (respectively, second countable).
   (b) Show that a product of countably many first countable (respectively, second countable) spaces is also first countable (respectively, second countable).
8. We proved the following results for metric spaces. Verify that the same proof will imply these results for any first-countable topological space.
   (a) **Theorem.** Let X be a topological space, and $A \subseteq X$. If there is a sequence of points in A converging to x, then $x \in \overline{A}$. If X is first-countable, then the converse holds: for every point $x \in \overline{A}$, there is some sequence of points in A converging to x.
   (b) **Theorem.** Let $f : X \to Y$ be a function of topological spaces. If $(a_n)_{n\in\mathbb{N}}$ is a sequence of points in X converging to a, then the sequence $(f(a_n))_{n\in\mathbb{N}}$ in Y converges to $f(a)$. If X is first-countable, then the converse holds: if for every sequence $(a_n)_{n\in\mathbb{N}}$ in X converging to a the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(a)$, then f is continuous.
9. Show that $\mathbb{R}^\omega$ (with the box topology) is not first-countable.
10. Let $Y \subseteq X$ be a subspace of a space X. Prove or disprove: if A is dense in X, then $A \cap Y$ is dense in Y.
11. For fixed $n \geq 0$, show that $\mathbb{R}^n$ is *separable* (Problem #4) by finding a countable dense subset.

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. **Definition (The Cantor set).** Let $A_0 = [0,1]$. Construct A_1 from A_0 be deleting the "middle third", the open set $(\frac{1}{3}, \frac{2}{3})$. Construct A_2 from A_1 by deleting the "middle thirds" $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$. More generally, define

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

The *Cantor set* is the set $C = \bigcap_{n \in \mathbb{N}} A_n$ as a subspace of the unit interval $[0,1]$.

   (a) Show that C is *totally disconnected*, that is, show that its connected components are single points.
   (b) Show that C is compact.
   (c) Show that C has no isolated points.
   (d) Show that C is uncountable.

2. **Definition (Uniform continuity).** A function $f : (X, d_X) \to (Y, d_Y)$ of metric spaces is *uniformly continuous* if for every $\epsilon > 0$, there is some $\delta > 0$ so that $d_Y(f(a), f(b)) < \epsilon$ for all points $a, b \in X$ with $d_X(a,b) < \delta$.

   In other words, this is a stronger notion of continuity, where the choice of δ does not depend on the point a.

   Prove the following theorem. *Hint:* Lebesgue numbers.

   Theorem (Uniform continuity theorem). Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous function from a compact metric space (X, d_X) to a metric space (Y, d_Y). Then f is uniformly continuous.

3. Given a Hausdorff and locally compact space X, our goal is to embed X into a compact Hausdorff space.

   Definition (Alexandroff compactification). Let X be a topological space, and let $\hat{X} = X \cup \{\infty\}$ be the set we obtain by adding a point ∞ which we call *the point at infinity.* We endow $\hat{X}$ with the following topology:

$$\mathcal{T} = \{ U \subseteq \hat{X} \mid U \subseteq X \text{ is open, or } \hat{X} \setminus U \text{ is a closed, compact subset of } X \}.$$

   The resulting topological space $\hat{X}$ is called the *Alexandroff extension* of X. If X is a noncompact, locally compact, Hausdorff space, then $\hat{X}$ is called the *one-point compactification* or the *Alexandroff compactification* of X.

   Notice, in particular, that X is an open subset of $\hat{X}$.

   (a) Verify that $\mathcal{T}$ is in fact a topology.
   (b) Verify that $\hat{X}$ is compact.
   (c) Suppose that X is locally compact and Hausdorff. Verify that $\hat{X}$ is Hausdorff.
   (d) Show that the inclusion map $X \hookrightarrow \hat{X}$ is an embedding.

(e) Show that, if X is not compact, then the closure of X (as a subset of $\hat{X}$) is $\hat{X}$. Show, in contrast, that if X is compact, then $X \cup \{\infty\}$ is a separation of $\hat{X}$.

(f) Suppose that X is noncompact, locally compact and Hausdorff. Show that any compactification of X by one point must be homeomorphic to the Alexandroff compactification.

(g) Suppose that a topological space X is a subspace of a compact Hausdorff space Y such that $Y \setminus X$ is a single point. Show that X must be Hausdorff and locally compact.

4. **Definition (Separable space).** A space X is called *separable* if X contains a countable dense subset A.

Note that the term "separable" is unrelated to the concept of a "separation" of X.

(a) Suppose that a space X is second countable. Show that X is separable.

(b) Show by example that a separable space need not be second countable.

(c) Show that a metric space X is second countable if and only if it is separable.

5. **Bonus (Optional)**.

Definition (Hausdorff dimension). Let X be a closed and bounded subset of $\mathbb{R}^n$. For $\delta > 0$, define $H^\delta_\epsilon(X)$ to be the infemum of the quantity $\sum_i \operatorname{diam}(U_i)^\delta$ taken over all open covers $\mathcal{U} = \{U_i\}$ of X with $\operatorname{diam}(U_i) < \epsilon$ for all i. Define

$$H^\delta(X) = \lim_{\epsilon \to 0} H^\delta_\epsilon(X).$$

The *Hausdorff dimension* $\dim_H(X)$ is defined to be the quantity

$$\dim_H(X) = \inf\{\delta \mid H^\delta(X) = 0\}.$$

(a) Show that a countable subset $X \subseteq \mathbb{R}^n$ has Hausdorff dimension 0.

(b) Compute the Hausdorff dimension of $[0,1]^m \times \{0\}^{n-m}$.

(c) Compute the Hausdorff dimension of the Cantor set C.