Recommended reading: Munkres Section 30–32.

Roughly similar content:
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Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Show that the following spaces are first countable.
   (i) any discrete space
   (ii) any indiscrete space
   (iii) $\mathbb{R}^n$
   (iv) $\mathbb{R}^\omega$ with the product topology

   (b) Show that the following spaces are not first countable.
   (i) $\mathbb{R}$ with the cofinite topology
   (ii) $\mathbb{R}$ with the cocountable topology

2. (a) Let X be a finite set. Show that any topology on X must be first and second countable.
   Note: A countable space X need not be first or second countable.

   (b) Show that a discrete space X is second countable if and only if the set X is countable.

3. (a) Suppose that X is a T_1-space. Show that, if X is regular, than X is Hausdorff.

   (b) Some authors do not require that regular spaces be T_1-spaces. Let $X = \{0, 1, 2, 3\}$ be the topological space with the topology $\mathcal{T} = \{\varnothing, \{0, 1\}, \{2, 3\}, X\}$; this space is not T_1. Show that X is regular (minus the T_1 condition), but X is not Hausdorff.

4. (a) Suppose that X is a T_1-space. Show that, if X is normal, than X is regular.

   (b) Some authors do not require that normal spaces be T_1-spaces. Let $X = \{0, 1\}$ be the topological space with the topology $\mathcal{T} = \{\varnothing, \{0\}, \{0, 1\}\}$; this space is not T_1. Show that X is normal (minus the T_1 condition), but X is not regular.

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. (a) Let X be a compact space, and let $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ be a nested sequence of nonempty closed subsets of X. Prove that their intersection $\bigcap_{n\in\mathbb{N}} C_n$ is nonempty.

   (b) **Definition (Baire space).** A space X is called a *Baire space* if the following condition holds. Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of closed subsets of X which each have empty interior. Then their union $\bigcup_{n\in\mathbb{N}} A_n$ has empty interior.

   Show that any discrete topological space is a Baire space, but $\mathbb{Q}$ is not.

   (c) Show the following result.

   Lemma (Equivalent definition of a Baire space). A space X is a Baire space if and only if, given any countable collection $\{U_n\}_{n\in\mathbb{N}}$ of open sets that are each dense in X, their intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.

   (d) Prove the following.

   Theorem (Baire category theorem for compact Hausdorff spaces). Any compact Hausdorff space X is a Baire space.

2. **Definition (Lindelöf space).** A space X is called *Lindelöf* if every open cover of X has a countable subcover.

   (a) Suppose that a space X is second countable. Show that X is Lindelöf.

   (b) Let X be a space with basis $\mathcal{B}$. Show that, to prove that every open cover of X has a countable (respectively, finite) subcover, it suffices to show that every cover by basis elements has a countable (respectivley, finite) subcover.

   (c) Show that the *Sogenfrey line* $\mathbb{R}_\ell$ is Lindelöf. Conclude in particular that a Lindelöf space need not be second countable.

   (d) Show that a metrizable space is Lindelöf if and only if it is second countable.

3. Prove the following results.

   (a) **Proposition (Equivalent definition of normal space).** Let X be a T_1-space. Then X is normal if and only if, for every closed subset $C \subseteq X$ and open set U containing C, there is an open set V containing C with $\overline{V} \subseteq U$.

   (b) In a sentence, describe how to adapt your proof in part (a) to show the folloiwng.

   Proposition (Equivalent definition of regular space). Let X be a T_1-space. Then X is regular if and only if, for every point $x \in X$ and neighbourhood U of x, there is a neighbourhood V of x with $\overline{V} \subseteq U$.

   (c) **Proposition (Subspaces of a regular space).** A subspace of a regular space is regular.

   (d) **Proposition (Subspaces of a normal space).** A **closed** subspace of a normal space is normal.

   (e) **Proposition (Products of regular spaces).** A product of regular spaces is regular.

   We have seen in class that a product of normal spaces need not be normal.

   (f) **Proposition (Neighbourhoods in normal spaces).** Let X be a normal space. Then every pair of distinct points in X have neighbourhoods whose closures are disjoint.

   (g) **Theorem (Metric spaces are normal).** Every metric space is normal.

   (h) **Theorem (Compact Hausdorff spaces are normal).** Every compact Hausdorff space is normal.

4. (a) Give an example of a continuous surjective map $p : X \to Y$ between T_1 spaces with the property that X is normal (and hence regular), but Y is neither normal nor regular.

   (b) Let $p : X \to Y$ be a continuous, closed, surjective map. Show that, if X is normal, then Y is normal.

5. **Bonus (Optional).** Prove the following theorem.

   Theorem (Urysohn metrization theorem). If X is a regular, second countable space, then X is metrizable.

   You are welcome to consult outside sources for this proof (such as Munkres Sections 34–35), but be sure to write it up comprehensively and in your own words.