**Recommended reading: Munkres Section 43, 45, 46.**

Roughly similar content:
https://faculty.etsu.edu/gardnerr/5357/notes/Munkres-43.pdf
https://faculty.etsu.edu/gardnerr/5357/notes/Munkres-45.pdf
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## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Prove that a closed subset $A$ of a complete metric space $(X, d)$ is also complete as a metric space $(A, d|_A)$ with respect to the restriction of the metric.
2. Let $(X, d)$ be a metric space, and let $\overline{d}(x, y) = \min\{d(x, y), 1\}$.
   (a) Verify that $\overline{d}$ is a metric on $X$.
   (b) Verify that $\overline{d}$ is equivalent to $d$. Conclude that a sequence converges in $(X, d)$ if and only if it converges in $(X, \overline{d})$.
   (c) Show that a sequence is Cauchy in $(X, d)$ if and only if it is Cauchy in $(X, \overline{d})$.
   (d) Conclude that $(X, d)$ is complete if and only if $(X, \overline{d})$ is complete.
3. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space $X$. Prove that $\{x_n \mid n \in \mathbb{N}\}$ is bounded.
4. Let $X$ be a metric space with the discrete metric. Show that $X$ is complete.
5. Let $X$ be a totally bounded metric space. Prove that $X$ is bounded.
6. (a) Let $X$ be a finite metric space. Show that $X$ is totally bounded.
   (b) Let $X$ be a compact metric space. Show that $X$ is totally bounded.
7. Let $Y^X$ denote the space of functions $f : X \to Y$ for some topological space $Y$ and set $X$.
   (a) Verify that the sup metric $\rho$ is well-defined on subset of bounded functions $f : X \to Y$.
   (b) Verify that, if $X$ is compact, then every function in $\mathscr{C}(X, Y)$ is bounded, so $\rho$ is well-defined on $\mathscr{C}(X, Y)$.
   (c) Verify that the uniform metric $\overline{\rho}$ satisfies
   $$\overline{\rho}(f, g) = \min\{\rho(f, g), 1\}.$$
   (d) Verift that both $\overline{\rho}$ and $\rho$ induce the uniform topology.
8. Let $\mathcal{F} \subseteq \mathscr{C}(X, Y)$. Show that if $\mathcal{F}$ is finite, then it is equicontinuous.

## Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. **Definition (Completion).** Let $X$ be a metric space. The *completion* of $X$ is a complete metric space $Y$ along with an isometric embedding $h : X \to Y$ such that $h(X)$ is dense in $Y$.

   In this question, we will construct the completion of $X$, and verify that it is unique up to isometry.
   (a) Let $A$ be a dense subset of metric space $Z$. Show that, if every Cauchy subsequence in $A$ converges in $Z$, then $Z$ is complete.

(b) Let $(X, d)$ be a metric space. Let $\tilde{X}$ denote the set of Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ in $X$. Let $Y$ denote the equivalences classes defined by the equivalence relation on $\tilde{X}$,

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \qquad \iff \qquad d(x_n, y_n) \xrightarrow{n\to\infty} 0.$$

Verify that this is indeed an equivalence relation.

(c) For $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, let $[\mathbf{x}]$ and $[\mathbf{y}]$ denote the corresponding equivalence classes. Define

$$D : Y \times Y \to \mathbb{R}$$
$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

Show that $D$ is well-defined, that is, its value does not depend on the choice of representative of the equivalence class.

(d) Show that $D$ defines a metric on $Y$.

(e) Define

$$h : X \to Y$$
$$x \mapsto [(x)_{n\in\mathbb{N}}]$$

sending a point $x$ to the constant sequence $(x)_{n\in\mathbb{N}}$. Show that $h$ is an isometric embedding.

(f) Show that, for any Cauchy sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in $X$, the sequence $(h(x_n))_{n\in\mathbb{N}}$ converges in $Y$ to $[\mathbf{x}]$. Conclude that $h(X)$ is dense ins $Y$.

(g) Further conclude that every Cauchy sequence in $h(X)$ must converge in $Y$, and thus by part (a) the space $(Y, D)$ is complete. This shows that $Y$ is the completion of $X$.

(h) Show that this completion is unique, in the sense of the following theorem.

**Theorem (The completion of $X$ is unique up to isometry).** Let $h : X \to Y$ and $h' : X \to Y'$ be isometric embeddings of the metric space $(X, d)$ into complete metric spaces $(Y, D)$ and $(Y', D')$, respectively. Then there is an isometry of $(\overline{h(X)}, D)$ and $(\overline{h'(X)}, D')$ that equals $h' \circ h^{-1}$ when restricted to the subspace $h(X)$.

2. (a) Prove the following theorem.

**Theorem (Compact metric spaces).** Let $(X, d)$ be a metric space. Then $X$ is compact if and only if it is complete and totally bounded.

(b) Let $X$ be a space and let $Y$ be a metric space. Suppose that $\mathcal{F} \subseteq \mathscr{C}(X, Y)$ is totally bounded in the uniform metric. Show that $\mathcal{F}$ is equicontinuous.

(c) Let $X$ be a compact space, and consider $\mathbb{R}^n$ with the Euclidean metric $d$. Show that any compact subset of $\mathscr{C}(X, \mathbb{R}^n)$ in the uniform topology is equicontinuous and pointwise bounded under $d$.

(d) Let $X$ be a compact space, and consider $\mathbb{R}^n$ with the Euclidean metric $d$. Let $\mathcal{F} \subseteq \mathscr{C}(X, \mathbb{R}^n)$ be equicontinuous and pointwise bounded under $d$. Show that its closure $\overline{\mathcal{F}}$ in the uniform topology is also equicontinuous and pointwise bounded under $d$.

(e) Let $X$ be a compact space, and let $Y$ be a compact metric space. Let $\mathcal{F} \subseteq \mathscr{C}(X, Y)$ be equicontinuous. Show that $\mathcal{F}$ is totally bounded in the uniform metric and in the sup metric.

(f) Let $X$ be a compact space, and consider $\mathbb{R}^n$ with the Euclidean metric $d$. Let $\mathcal{F} \subseteq \mathscr{C}(X, \mathbb{R}^n)$ be equicontinuous and pointwise bounded under $d$. Show that there exists a compact subset $Y \subseteq \mathbb{R}^n$ such that $\bigcup_{f\in\mathcal{F}} f(X)$ is contained in $Y$.

(g) Prove the following theorem.

**Theorem (Ascoli's Theorem).** Let $X$ be a compact space, and consider $\mathbb{R}^n$ with the Euclidean metric $d$. A subset $\mathcal{F} \subseteq (X, \mathbb{R}^n)$ has compact closure (in the uniform topology) if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under $d$.

(h) Prove the following theorem.

**Theorem (Arzela's Theorem).** Let $X$ be a compact space, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathscr{C}(X, \mathbb{R}^k)$. If $\{f_n \mid n \in \mathbb{N}\}$ is pointwise bounded and equicontinuous, then the sequence has a uniformly convergent subsequence.

3. (a) Let $X$ be a compact space, and let $Y$ be a metric space. Show that the uniform topology, the topology of compact convergence, and the compact-open topology on $\mathscr{C}(X, Y)$ all coincide. In particular, conclude that these topologies depend only on the topology on $Y$, and not the particular metric.

(b) Prove the following theorem.

**Theorem (The evaluation map is continuous).** Let $X$ be a locally compact, Hausdorff space. Let $Y$ be a space, and consider $\mathscr{C}(X, Y)$ with the compact-open topology. The evaluation map $e$ is continuous:

$$\begin{aligned} e : X \times \mathscr{C}(X, Y) &\to Y \\ e(x, f) &= f(x). \end{aligned}$$

(c) Prove the following theorem.

**Theorem (The product-Hom adjunction).** Let $X$ and $Y$ be spaces, and $\mathscr{C}(X, Y)$ have the compact-open topology. If $f : X \times Z \to Y$ is a continuous map, then the induced map

$$\begin{aligned} F : Z &\to \mathscr{C}(X, Y) \\ z &\mapsto [x \mapsto f(x, z)] \end{aligned}$$

is continuous. The converse holds if $X$ is locally compact and Hausdorff.

4. (a) Let $X$ be a complete metric space, and let $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ be a nested sequence of nonempty closed subsets of $X$. Show that, if $(\operatorname{diam}(C_n))_{n\in\mathbb{N}}$ converges to zero, then $\bigcap_{n\in\mathbb{N}} C_n$ is nonempty.

(b) Prove the following.

**Theorem (Baire category theorem for complete metric spaces).** Any complete metric space $X$ is a Baire space.