Recommended background reading: Munkres Sections 1–6. These sections are assumed background in the class – if you feel that you would benefit from reviewing this material, I strongly recommend taking the time to do so over the next couple weeks.

Munkres is recommend, but if you have elected not to purchase the textbook, then the following online notes have roughly similar content to these first 6 sections:
`http://math.mit.edu/~apm/cha.pdf`,
`https://www.math.uh.edu/~dlabate/settheory_Ashlock.pdf`

This week's recommended reading: Munkres Sections 7, 9.
Roughly similar content:
`https://en.wikipedia.org/wiki/Countable_set`, up to and including the section "Some technical detail"
`http://mathworld.wolfram.com/AxiomofChoice.html`

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X be a set and $A, B, C \subseteq X$ be subsets.
   (a) Show by example that $A \cap (B \cup C)$ need not equal $(A \cap B) \cup C$.
   (b) Show that
   $$A\cap(B\cup C) = (A\cap B)\cup(A\cap C) \qquad A\cup(B\cap C) = (A\cup B)\cap(A\cup C) \qquad A = (A\cap B)\cup(A\setminus B).$$
   (c) Prove DeMorgan's Laws,
   $$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \qquad A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$
2. Show that
   $$A \times (B \setminus C) = (A \times B) \setminus (A \times C).$$
3. Determine which of the following subsets of of $\mathbb{R}^2$ can be expressed as the Cartesian product of two subsets of $\mathbb{R}$.
   (a) $\{(x, y) \mid x \in \mathbb{Q}\}$ (b) $\{(x, y) \mid x > y\}$ (c) $\{(x, y) \mid 0 < y \leq 1\}$ (d) $\{(x, y) \mid x^2 + y^2 < 1\}$
4. Let A, B be sets. If $A \times B$ is finite, does it follow that A and B are finite? *Hint*: Mind the empty set!
5. If A and B are finite sets, show that the set of all functions $f : A \to B$ is also finite.
6. Let $f : X \to Y$ and $g : Y \to Z$ be functions.
   (a) Show that if f and g are both injective, then so is $g \circ f : X \to Z$.
   (b) Show that if f and g are both surjective, then so is $g \circ f : X \to Z$.
   (c) Show by example that if only one of g and f injects (respectively, surjects), then $g \circ f$ may not.
   (d) Show by example that if $g \circ f$ is injective (respectively, surjective), then f and g need not both be.
7. (a) Let $f : X \to Y$ be a function that is both injective and surjective. Prove that f is *invertible*, that is, there is a well-defined function $f^{-1} : Y \to X$ such that $f^{-1} \circ f$ is the identity function on X, and $f \circ f^{-1}$ is the identity function on Y.
   (b) Prove conversely that every invertible function is both injective and surjective.
   (c) Prove that, if f is invertible, then its inverse f^{-1} is unique.
   (d) Let $f : X \to Y$ be an invertible function, and let $A \subseteq Y$. Show that $f^{-1}(A)$ denotes the same set, whether it is interpreted as the preimage of A under f, or the image of A under the inverse function $f^{-1} : Y \to X$. This exercise shows that our notation conventions are compatible.

   Remark: The preimage $f^{-1}(A)$ is a well-defined set whether or not the function f is invertible.

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. **(Review of one-sided inverses.)** Let A be a set. Recall that the *identity function* i_A on A is the map

$$i_A : A \longrightarrow A$$
$$i_A(a) = a.$$

Let $f : X \to Y$ be a function. Recall that a function $g : Y \to X$ is a *left inverse* of f if $g \circ f = i_X$, and a function $h : Y \to X$ is a *right inverse* to f if $f \circ h = i_Y$.

   (a) Show that if f has a right inverse, then f is surjective, and if f has a left inverse, then f is injective.

   (b) Give an example of a function with a left inverse, but no right inverse.

   (c) Give an example of a function with a right inverse, but no left inverse.

   (d) Let $f : X \to Y$ be a function. Can f have more than one left inverse? More than one right inverse?

   (e) Show that if f has a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$.

2. **(Equivalence relations review.)** Review the concept of a *binary relation*, an *equivalence relation*, *equivalence classes*, and a *partition* of a set. You may do this by referring to Section 3 of Munkres, or (for example) by following the appropriate links on Wikipedia.

   (a) State, in your own words, the definition of an *equivalence relation* on a set X. Explain in your own words why the equivalence classes define a partition of X.

   (b) Let $f : X \to Y$ be a function. Show that X is partitioned into preimages of elements of Y. In other words, show that the relation

$$x_1 \sim x_2 \qquad \Longleftrightarrow \qquad f(x_1) = f(x_2)$$

   is an equivalence relation, with corresponding equivalence classes $\left\{ f^{-1}\big(\{y\}\big) \;\middle|\; y \in f(X) \right\}$.

   Remark: Notice that, in contrast, $\left\{ f\big(\{x\}\big) \;\middle|\; x \in X \right\}$ is not necessarily a partition of Y.

   (c) Let X be any set. For $A, B \subseteq X$, give a complete and rigorous proof that the relation

$$A \sim B \qquad \Longleftrightarrow \qquad |A| = |B|$$

   is an equivalence relation on the set of subsets of X.[1]

3. **(Induction and the well-ordering property).**

   (a) Prove by induction that, for any $n \in \mathbb{N}$, every nonempty subset of the set $\{1, 2, \ldots, n\}$ contains a smallest element.

   (b) Deduce from part (a) that every nonempty subset of $\mathbb{N}$ has a smallest element. This is called the *well-ordering property* of $\mathbb{N}$.

   (c) Prove by induction that, for any $n \in \mathbb{N}$, every nonempty subset of $\{1, 2, \ldots, n\}$ contains a largest element.

   (d) Explain why you cannot conclude from part (c) that every nonempty subset of $\mathbb{N}$ has a largest element.

[1] We will run into set-theoretic issues if we try to formulate the statement that cardinality defines an equivalence relation on "all possible sets", since the collection of "all possible sets" is somehow too big to itself be a set. These issues are beyond the scope of this class.

Henceforth, you can take elementary statements about $\mathbb{N}$ for granted. Here are some examples of statements that you can invoke without proof:

- Every nonempty subset of $\mathbb{N}$ has a least element.
- Every nonempty subset of $\mathbb{Z}$ that is bounded below contains a least element.
- Every nonempty subset of $\mathbb{Z}$ that is bounded above contains a largest element.
- The real numbers $\mathbb{R}$ have the *least upper bound property*: every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.
- The real numbers of the corresponding *greatest lower bound property.*

4. **(Power sets review.)** Let A be a set. Recall that the *power set* $\mathscr{P}(A)$ of A is the set of all subsets of A. For example,
$$\mathscr{P}\Big(\{0,1\}\Big) = \Big\{\varnothing, \{0\}, \{1\}, \{0,1\}\Big\}.$$
   (a) Suppose that A is a set of finite cardinality $|A| = n$. Prove that its power set has cardinality $|\mathscr{P}(A)| = 2^n$.
   (b) Let A be any set. Show that A has strictly smaller cardinality than $\mathscr{P}(A)$. That is, show that there exists an injective map $A \to \mathscr{P}(A)$, but no bijective map exists.

5. **(Countable and uncountable sets).** Determine (with justification) whether or not each of the following sets are countable.
   (a) The set of all functions $f : \{0,1\} \to \mathbb{N}$.
   (b) The set $\mathbb{N}^n$, for fixed $n \in \mathbb{N}$.
   (c) The set $\bigcup_n F_n$, where F_n is the set of functions $f : \{1, 2, \dots, n\} \to \mathbb{N}$.
   (d) The set of all functions $f : \mathbb{N} \to \{0,1\}$.
   (e) The set of all functions $f : \mathbb{N} \to \{0,1\}$ that are eventually 0. (This means that, for each f, there is some number N so that $f(n) = 0$ for all $n \geq N$.)
   (f) The set of all functions $f : \mathbb{N} \to \{0,1\}$ that are eventually constant.
   (g) The set of all 2-element subsets of $\mathbb{N}$.
   (h) The set of all finite subsets of $\mathbb{N}$.
   (i) The set $\mathscr{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$

6. **Bonus (Optional).** Prove the (surprisingly non-trivial!) Cantor–Schroeder–Bernstein Theorem.

   For sets A and B, we write $|A| \leq |B|$ if there exists an injective map $A \to B$.

   Theorem (Cantor–Schroeder–Bernstein). Let A and B be sets such that $|A| \leq |B|$ and $|B| \leq |A|$. Then $|A| = |B|$.

   In other words, this theorem states that, if there are injective maps $A \to B$ and $B \to A$, then there exists a bijection between A and B.

   Remark: This theorem is the main step in the proof that the binary relation $\leq$ defines a *partial order* on the *cardinal numbers.*