Recommended reading: Munkres Section 12 & Section 18 (to the end of Example 2).

Roughly similar content:
Hatcher `https://pi.math.cornell.edu/~hatcher/Top/TopNotes.pdf`,
subsections "Topological Spaces", "Basis for a Topology", and "Continuity and Homeomorphisms".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X be a set.
   (a) Let $\mathcal{T} = \{\varnothing, X\}$. Prove that $\mathcal{T}$ is a topology on X. It is called the *indiscrete topology.*
   (b) Show that $\mathcal{B} = \{X\}$ is a basis for the indiscrete topology on X.
   (c) Let $\mathcal{T}$ be the collection of all subsets of X. Prove that $\mathcal{T}$ is a topology on X, called the *discrete topology.*
   (d) Show that $\mathcal{B} = \left\{ \{x\} \;\middle|\; x \in X \right\}$ is a basis for the discrete topology on X.
2. Let $X = \{0, 1, 2\}$. Show that the collection of subsets $\Big\{\varnothing, X, \{0, 1\}, \{1, 2\}\Big\}$ is **not** a topology on X.
3. Verify that the standard topology on $\mathbb{R}$ is in fact a topology.
4. Let X be a set and let $p \in X$. Prove that $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X \mid p \in U\}$ is a topology on X.
5. Find all possible topologies on the set $X = \{0, 1\}$.
6. Let $X = \{a, b, c, d\}$ and let
$$\mathcal{T} = \Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\Big\}.$$
   (a) Verify that $\mathcal{T}$ is a topology on X.
   (b) Determine which of the following sets are bases for this topology.

      (i) $\mathcal{B} = \{\varnothing, X\}$
      (ii) $\mathcal{B} = \{\{a\}, \{b\}, \{c\}, \{d\}\}$
      (iii) $\mathcal{B} = \mathcal{T}$
      (iv) $\mathcal{B} = \{\{a\}, \{b\}, \{b, c\}, \{a, d\}\}$
      (v) $\mathcal{B} = \{\{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}$
      (vi) $\mathcal{B} = \{\{a, b\}, \{b, c\}, \{a, d\}\}$
7. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show by induction that the intersection of any **finite** collection of open sets is open.
   (b) Explain why this argument does not apply to an infinite collection of open sets.
8. See Assignment Problem #4 for the definition of the *cofinite* and *cocountable* topologies on a set X.
   (a) Suppose that X is a finite set. Show that the cofinite topology on X coincides with the discrete topology on X. Show that this is not the case when X is infinite.
   (b) Suppose that X is a countable set. Show that the cocountable topology coincides with the discrete topology. Show that this is not the case when X is uncountable.
9. Let $(X, \mathcal{T})$ be a topological space. Verify that X and $\varnothing$ are always closed subsets of X.
10. **(The openness criterion).** Let X be a topological space, and let $V \subseteq X$. Show that, to prove that V is open, it suffices to check the following: For every $x \in V$ there is some set $U_x \subseteq X$ containing x such that U_x is open and $U_x \subseteq V$. (This criterion will be very useful!)

11. (a) Let X be a topological space. Show that the identity map is continuous:

$$id_X : X \to X$$
$$id_X(x) = x$$

(b) Let X and Y be topological spaces, and fix $y_0 \in Y$. Show that the constant map is continuous:

$$f : X \to Y$$
$$f(x) = y_0$$

12. Prove the following result.

Proposition (Composition of continuous functions). Let X, Y, and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then the composition map $g \circ f : X \to Z$ is continuous.

13. (a) Let X be a topological space with the discrete topology. Show that any function $f : X \to Y$ for any topological space Y is necessarily continuous.
   (b) Let Y be a topological space with the indiscrete topology. Show that any function $f : X \to Y$ for any topological space X is necessarily continuous.

14. See Assignment Problem #5 for the definition of *coarser* and *finer* topologies.
   (a) Let X be a set. Show that the indiscrete topology on X is coarser than any other topology on X.
   (b) Let X be a set. Show that the discrete topology on X is finer than any other topology on X.

15. Give examples of two distinct topologies on the set $\{a, b, c\}$ where one is finer than the other, and an example of two topologies on the set $\{a, b, c\}$ that are not comparable (in the sense of finer/coarser).

16. Let X be a set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Show that the following statements are equivalent.

   - $\mathcal{T}_1 \subseteq \mathcal{T}_2$
   - The identity map $id_X : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous
   - The identity map $id_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is open

17. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show that $\mathcal{T}$ is a basis for $\mathcal{T}$.
   (b) Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Show that any collection of open sets in X containing $\mathcal{B}$ is also a basis for $\mathcal{T}$.

18. **(The basis criterion).** Let $(X, \mathcal{T})$ be a topological space. Show that a collection $\mathcal{B}$ of open sets is a basis and that $\mathcal{B}$ generates $\mathcal{T}$ if and only if it satisfies the following property: For every open set $U \in \mathcal{T}$, and every $x \in U$, there exists some element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

19. Show that the following collections of sets are bases for topologies on $\mathbb{R}$. For which of the pairs of topologies associated to these bases are comparable (in terms of being finer/coarser)?

   - $\mathcal{B}_1 = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}$
   - $\mathcal{B}_2 = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}$
   - $\mathcal{B}_3 = \{(a, \infty) \mid a \in \mathbb{R}\}$
   - $\mathcal{B}_4 = \{U \mid U \subseteq \mathbb{R},\ \mathbb{R} \setminus U \text{ is finite}\}$

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Recall the following definition from real analysis:

Definition (Convergence of sequences in $\mathbb{R}$). A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers converges to a point $a_\infty \in \mathbb{R}$ if, for each $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that $|a_n - a_\infty| < \epsilon$ for all $n \geq N$.

Prove the following result.

> **Proposition (Equivalent definition of convergence in $\mathbb{R}$).** A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers converges to $a_\infty \in \mathbb{R}$ if and only if it satisfies the following condition: for each open set $U \subseteq \mathbb{R}$ containing a_∞, there is some $N \in \mathbb{N}$ so that $a_n \in U$ for all $n \geq N$.

Remark: This shows that, (like continuous functions), convergence of sequences can be characterized entirely in terms of open sets. This concept will also make sense for abstract topological spaces.

2. **(Unions and intersections of closed subsets).** Consider the following definition.

   > **Definition (Closed subsets of a topological space).** Let $(X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is called *closed* if its complement is open, that is, if $X \setminus C$ is an element of $\mathcal{T}$.

   (a) Let $C_1, C_2, \ldots, C_n$ be a finite collection of closed sets. Show that their union $\bigcup_{i=1}^n C_i$ is closed.

   (b) Show by example that the union of an arbitrary number of closed subsets need not be closed.

   (c) Suppose that $\{C_i\}_{i\in I}$ is a collection of closed sets in X. Verify that $\bigcap_{i\in I} C_i$ is closed.

3. Prove the following.

   > **Proposition (Equivalent definition of continuity).** Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if it satisfies the following condition: for every closed set $C \subseteq Y$, the preimage $f^{-1}(C) \subseteq X$ is closed.

4. **(The cofinite and cocountable topologies).** Let X be a set.

   (a) **(The cofinite topology).** Let $\mathcal{T}_{cf}$ be the collection of subsets of X

   $$\mathcal{T}_{cf} = \{\varnothing\} \cup \{U \subseteq X \mid X \setminus U \text{ is a finite set}\}.$$

   Verify that $\mathcal{T}$ is a topology on X. It is called the *cofinite topology.*

   (b) **(The cocountable topology).** Let $\mathcal{T}_{cc}$ be the collection of subsets of X

   $$\mathcal{T}_{cc} = \{\varnothing\} \cup \{U \subseteq X \mid X \setminus U \text{ is a countable set}\}.$$

   It is called the *cocountable topology.* In a few sentences, explain how you could modify your argument from part (a) to prove that $\mathcal{T}_{cc}$ is a topology.

5. **(Coarser and finer topologies).** Consider the following definitions.

   > **Definition (Coarser topology; finer topology).** Let X be a set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then the topology $\mathcal{T}_1$ is said to be *coarser* than $\mathcal{T}_2$, and the topology $\mathcal{T}_2$ is said to be *finer* than the topology $\mathcal{T}_1$.

   Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a continuous function. Show that f will still be continuous if we replace $\mathcal{T}_X$ by any finer topology on X, or if we replace $\mathcal{T}_Y$ with any coarser topology on Y.

6. An advantage of identifying a basis for a topology is that many topological statements can be reduced to statements about the basis. As an example, prove the following result.

   > **Proposition (Equivalent definition of continuity).** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}_Y$ be a basis for $\mathcal{T}_Y$. Prove that a map $f : X \to Y$ is continuous if and only if for every open set $U \in \mathcal{B}_Y$, the preimage $f^{-1}(U) \subseteq X$ is open.

7. (a) Show that $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis and that it generates the standard topology on $\mathbb{R}$.

   (b) Show that $\mathcal{C} = \{[a, b) \mid a, b \in \mathbb{Q}\}$ is a basis and that it generates a topology different from the *lower limit topology* (which has basis $\{[a, b) \mid a, b \in \mathbb{R}\}$).