Recommended reading: Munkres Section 13, 14, 15, 16, 17.

Roughly similar content:
Hatcher `https://pi.math.cornell.edu/~hatcher/Top/TopNotes.pdf`,
subsections "Subspaces", "Product spaces", & "Interior, Closure, and Boundary"
Warning: Hatcher uses a different definition of *limit point* than we (or Munkres) do.
`https://en.wikipedia.org/wiki/Order_topology`

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Show by example that topological spaces X can have subsets that are
   - closed and not open
   - open and not closed
   - both closed and open
   - neither closed nor open

   (b) Recite the Topologist Scout Oath:

   "On my honour, I will do my best
   to never claim to prove a set is closed by showing that it is not open,
   and to never claim to prove a set is open by showing that it is not closed."

2. (a) Let X be a topological space with the discrete topology. Show that every subset of X is closed.

   (b) Let X be a topological space with the indiscrete topology. Show that the closed subsets of X are precisely $\{\varnothing, X\}$.

   (c) Let X be a topological space with the cofinite topology. Show that the closed sets are precisely the sets $\{X\} \cup \{S \subseteq X \mid S \text{ is finite}\}$.

3. See Assignment Question #3 for the definition of the *subspace topology*. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X
$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}.$$
   Write down the subspace topology $\mathcal{T}_S$ induced on the subset $S = \{a, b, d\} \subseteq X$.

4. Consider the set $Y = (0, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

   (a) $A = \{\, x \mid \frac{1}{2} < |x| < 1\}$
   (b) $A = \{\, x \mid \frac{1}{2} \leq |x| < 1\}$
   (c) $A = \{\, x \mid \frac{1}{2} < |x| \leq 1\}$
   (d) $A = \{\, x \mid \frac{1}{2} \leq |x| \leq 1\}$
   (e) $A = \{\, \frac{1}{n} \mid n \in \mathbb{N}\}$

5. (a) Suppose that X is a topological space with the discrete topology. Is the subspace topology of a subset $S \subseteq X$ necessarily the discrete topology on S?

   (b) Suppose that X is a topological space with the indiscrete topology. Is the subspace topology of a subset $S \subseteq X$ necessarily the indiscrete topology on S?

6. (a) Let $(X, \mathcal{T})$ be a topological space. Let $S \subseteq X$ and let $\mathcal{T}_S$ be the subspace topology on S. Prove that if S is an open subset of X, and if $U \in \mathcal{T}_S$, then $U \in \mathcal{T}$.

   (b) Suppose that X is a topological space and $S \subseteq X$. Show that, if S is open, then the inclusion map i_S (See Assignment Question #3) is an open map.

7. Let X be a topological space. Let $A \subseteq B \subseteq X$, and let B be a topological space with the subspace topology. Show that the subspace topology on A as a subspace of X coincide with the subspace topology on A as a subspace of X.

8. See Assignment Problem #4 for the definition of the *order topology* on a totally ordered set X. Consider the rays $(a, \infty) = \{x \in X \mid a < x\} \subseteq X$. Show that the collections of sets $\{(a, \infty) \mid a \in X\}$ are the basis for a topology on X, and moreover that this topology is coarser than the order topology.

9. Show that the order topology on $\mathbb{Z}$ is the discrete topology.

10. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $y_0 \in Y$. Consider the map

$$\begin{aligned} f : X &\longrightarrow X \times Y \\ x &\longmapsto (x, y_0) \end{aligned}$$

where $X \times Y$ has the product topology. Show that the map f is always continuous, but that it may not be an open map.

11. Let X be a topological space, and $A \subseteq X$ Explain the sense in which $\overline{A}$ is the "smallest" closed set containing A, and the sense in which $\text{int}(A)$ is the "largest" open set contained in A.

12. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

Find the interior and closure of the subsets

(a) $\{a, b, c\}$ (b) $\{a, c, d\}$ (c) $\{a, b, d\}$ (d) $\{b\}$ (e) $\{d\}$ (f) $\{b, d\}$

13. Show the following.
    (a) If $A \subseteq B$, then $\text{Int}(A) \subseteq \text{Int}(B)$.
    (b) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
    (c) If A is closed in Y and Y is closed in X, then A is closed in X.

14. Find all limit points of the following subsets of $\mathbb{R}$ (with the standard topology).

(a) $\mathbb{R}$ (b) $\varnothing$ (c) $(0, 1)$ (d) $\{0\}$ (e) $\mathbb{N}$ (f) $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ (g) $\mathbb{Q}$ (h) $\left\{\frac{a}{2^n} \mid a, n \in \mathbb{N}\right\}$

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. (a) Let X and Y be topological spaces. Prove or give a counterexample: if $f : X \to Y$ is an open map, then $f(C)$ is closed for every closed set $C \subseteq X$.
   (b) State examples of the following. No justification necessary.
      (i) topological spaces X and Y, and a map $f : X \to Y$ that is open but not continuous,
      (ii) topological spaces X and Y, and a map $f : X \to Y$ that is continuous but not open,
      (iii) topological spaces X and Y, and a map $f : X \to Y$ that is both open and continuous,
      (iv) topological spaces X and Y, and a map $f : X \to Y$ that is neither open nor continuous.

2. **Definition (Subbases).** Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X whose union is equal to X. Then the *topology generated by the subbasis* $\mathcal{S}$ is the collection of all arbitrary unions of all finite intersections of elements in $\mathcal{S}$.

   Remark: Notably, in contrast to a basis, we are permitted to take finite intersections of sets in a subbasis.

(a) Show that the set $\mathcal{T}$ generated by a subbasis $\mathcal{S}$ is a topology, and is moreover the coarsest topology containing $\mathcal{S}$. *Hint:* It suffices to show that the collection of all finite intersections of elements of $\mathcal{S}$ is a basis.

(b) Verify that $\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, a) \mid a \in \mathbb{R}\}$ is a subbasis for the standard topology on $\mathbb{R}$.

(c) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Verify that

$$\mathcal{S} = \{\pi_X^{-1}(U) \mid U \in \mathcal{T}_X\} \cup \{\pi_Y^{-1}(V) \mid V \in \mathcal{T}_Y\}$$

is a subbasis for the product topology on $X \times Y$. Conclude that the product topology is the coarsest topology making the projection maps π_X and π_Y continuous.

3. Given a topological space X, there is an induced topological space structure on any subset $S \subseteq X$.

Definition (The subspace topology). Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be a subset. Then the *subspace topology* on S is defined to be

$$\mathcal{T}_S = \{ \ U \cap S \mid U \in \mathcal{T}_X \ \}.$$

(a) Verify that the subspace topology $\mathcal{T}_S$ is in fact a topology.

(b) Show that a set $C \subseteq S$ is closed if and only if there is some set $D \subseteq X$ that is closed with $C = D \cap S$.

(c) Define the *inclusion* of S into X to be the map

$$i_S : S \to X$$
$$i_S(s) = s.$$

Show that this map is continuous with respect to the topology $\mathcal{T}_X$ on X and the subspace topology $\mathcal{T}_S$ on S. Show moreover that $\mathcal{T}_S$ is precisely the set $\{i_S^{-1}(U) \mid U \subseteq X \text{ open}\}$, so the subspace topology is the coarsest possible topology on S making the inclusion map continuous.

(d) Suppose that S is an **open** set in the topological space X. Show that the map i is an open map. (Recall that this means that $i(V) \subseteq X$ is open for every open set $V \subseteq S$.)

(e) Let $\mathcal{B}_X$ be a basis for the topology on X. Show that the set $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}_X\}$ is a basis for the subspace topology on S.

4. **(The order topology).**

(a) Review the definition of a *totally ordered set* (also called a *simply* or *linearly* ordered set); either in Munkres Section 3, or (eg) on Wikipedia. State, in your own words, the definition of a total order on a set.

(b) **Definition (The order topology).** Let X be a set with at least two elements and a total order $<$. Then the *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of the following sets:

- All open intervals $(a, b) = \{x \mid a < x < b\}$ in X.
- All half-open intervals $[a_0, b) = \{x \mid a_0 \leq x < b\}$, where a_0 is the smallest element (if any) of X.
- All half-open intervals $(a, b_0] = \{x \mid a \leq x < b_0\}$, where b_0 is the largest element (if any) of X.

Prove that the set $\mathcal{B}$ is in fact a basis.
Remark: Notice that the order topology on $\mathbb{R}$ (with its usual order) is the standard topology.

(c) Give an example (with proof) of a subset $S \subseteq \mathbb{R}$ where the subspace topology on S (as a subset of $\mathbb{R}$) is **not** the same as the order topology on S (with the order inherited from the order on $\mathbb{R}$).

(d) A subset Y of an ordered set X is called *convex* if, whenever a and b are elements of Y, the whole interval $(a, b) \subseteq Y$. Prove that, if Y is a convex subset of X, then order topology on Y coincides with the subspace topology on Y induced by the order topology on X.

5. Let X and Y be topological spaces. Suppose that $\mathcal{B}_X$ is a basis generating the topology on X, and $\mathcal{B}_Y$ for Y. Show that the set
$$\{C \times D \mid C \in \mathcal{B}_X, D \in \mathcal{B}_Y\}$$
is a basis for the product topology on $X \times Y$.

6. Let X be a topological space, and let $A \subseteq X$.

   (a) Show that A is closed if and only if $A = \overline{A}$.

   (b) Show that $\overline{\overline{A}} = \overline{A}$.

   (c) Show that A is open if and only if $A = \text{Int}(A)$.

   (d) Show that $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.

7. Let X be a topological space with basis $\mathcal{B}$, and $A \subseteq X$.

   (a) Show that $x \in \overline{A}$ if and only if every **basis element** $B \in \mathcal{B}$ containing x intersects A.

   (b) Show that $x \in \text{Int}(A)$ if and only if there exists a **basis element** $B \in \mathcal{B}$ with $x \in B \subseteq A$.

8. Let X be a topological space, and let $A \subseteq X$. Let A' be the set of all limits points of A. Prove that $\overline{A} = A \cup A'$. Conclude that a set A is closed if and only if it contains all of its limit points.

9. **Bonus (Optional)**. Consider the standard topology on $\mathbb{R}$, and let $\mathbb{R}^\omega$ denote the product
$$\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$$
of a countably infinite number of copies of $\mathbb{R}$. The space $\mathbb{R}^\omega$ has a topology given by the basis
$$\mathcal{B}_b = \{U_1 \times U_2 \times U_3 \times \cdots \mid U_i \subseteq \mathbb{R} \text{ open for all } i \in \mathbb{N}\}.$$
(You do not need to verify that $\mathcal{B}_b$ is basis). Show that there exist maps $f_i : \mathbb{R} \to \mathbb{R}$ for $i \in \mathbb{N}$ such that f_i is continuous for all i, but the map
$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^\omega \\ x &\longmapsto (f_1(x), f_2(x), f_3(x), \cdots) \end{aligned}$$
is not continuous. In part for this reason, this is usually not the preferred topology on $\mathbb{R}^\omega$.