Recommended reading: Munkres Section 17.

Roughly similar content: Definitions, Examples, Properties subsections of the following
https://en.wikipedia.org/wiki/Interior_(topology), https://en.wikipedia.org/wiki/Closure_(topology), https://en.wikipedia.org/wiki/Boundary_(topology), https://en.wikipedia.org/wiki/Hausdorff_space, https://en.wikipedia.org/wiki/Limit_of_a_sequence#Topological_spaces

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Consider the interval $(0,1]$ as a subset of $\mathbb{R}$ with the standard topology. Give a complete, rigorous proof that its closure is $[0,1]$, its interior is $(0,1)$, its boundary is $\{0,1\}$, and its set of limit points is $[0,1]$.

2. Consider $\mathbb{R}$ with the standard topology. Find examples of subsets A of $\mathbb{R}$ with the following properties.
   (a) $\partial A = \varnothing$.
   (b) $\partial A \neq \varnothing$, and $\partial A \subseteq A$.
   (c) $\partial A \neq \varnothing$, and $A \cap \partial A = \varnothing$.
   (d) $\partial A \neq \varnothing$, and $A = \partial A$.
   (e) A is a **proper** subset of ∂A.
   (f) $\partial(\partial A) \neq \partial A$.

3. Consider the following subsets of $\mathbb{R}$
   - $\mathbb{R}$
   - $\varnothing$
   - $\{0,1\}$
   - $[0,\infty)$
   - $(1,2)$
   - $[1,2] \cup [3,\infty)$
   - $(-\infty,0)$
   - $(-\infty,0]$
   - $\mathbb{N}$
   - $\{-n \mid n \in \mathbb{N}\}$

   Find the interior, closures, boundaries, and limit points of these subsets ...
   (a) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.
   (b) ... when $\mathbb{R}$ has the cofinite topology.
   (c) ... when $\mathbb{R}$ has the cocountable topology.
   (d) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.

4. Let X be a topological space and $A \subseteq X$. Show that $\overline{A} = A \cup \partial A = \mathrm{Int}(A) \cup \partial A$. Conclude that $\overline{A}$ is the disjoint union of $\mathrm{Int}(A)$ and ∂A.

5. (a) Consider $\mathbb{R}$ with the standard topology. Give a complete and rigorous proof that the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ converges to 0.
   (b) Consider the interval $(0,1)$ with the standard topology. Give a complete and rigorous proof that the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ of point in $(0,1)$ does not converge to any element of $(0,1)$.

6. Suppose that X is a topological space, and that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to $a_\infty \in X$. Prove that any subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to $a_\infty \in X$.

7. Show by example that the limit of a sequence of points in a topological space need not be unique.

8. Let X be a topological space and $x \in X$. Show that the constant sequence $(x)_{n\in\mathbb{N}}$ converges to x.

9. A sequence $(a_n)_{n\in\mathbb{N}}$ is called *eventually constant* if there is some $N \in \mathbb{N}$ so that $a_n = a_N$ for all $n \geq N$.
   (a) Let X be a set with the discrete topology. Show that the only convergent sequences are the sequences that are eventually constant. What is their set of limits?
   (b) Let X be a set with the indiscrete topology. Show that all sequences converge to every point in X.

10. Let $X = \{a,b,c,d\}$ be a set endowed with the topology

$$\mathcal{T} = \Big\{\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,c,d\}\Big\}.$$

    Find the set of all limits of each of the following sequences.

- $d, d, d, d, d, \ldots$
- $c, c, c, c, c, \ldots$
- $a, a, a, a, a, \ldots$
- $a, c, a, c, a, c, \ldots$
- $a, b, c, d, a, b, c, d, \ldots$
- $a, b, a, b, a, b, \ldots$

11. (a) Consider the following sequences of real numbers.
    - $0, 0, 0, 0, \cdots$
    - $1, 1, 1, 1, \cdots$
    - $0, 1, 0, 1, \cdots$
    - $(n)_{n \in \mathbb{N}}$
    - $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$

    Find the set of all limits for each of these sequences ...

    (b) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

    (c) ... when $\mathbb{R}$ has the cofinite topology.

    (d) ... when $\mathbb{R}$ has the cocountable topology.

    (e) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.

12. (a) Let X be a Hausdorff topological space, and $S \subseteq X$. Prove the subspace topology on S is Hausdorff.

    (b) Let X and Y be Hausdorff topological spaces. Prove the product topology on $X \times Y$ is Hausdorff.

    (c) Let X be a totally ordered set. Show that the order topology on X is Hausdorff.

13. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

    (a) Is X Hausdorff?

    (b) Find the interior and closure of $\{a, c, d\}$.

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. **(Finite unions, intersections, and products of interiors and closures).**

    (a) Let X a topological space, with $A, B \subseteq X$. For each of the following, determine whether you can replace the symbol $\Box$ with $\subseteq, \supseteq, =$, or none of the above. You should understand how to justify your solution, but for this step, you only need to submit the final answer ($\subseteq, \supseteq, =$, or "none of the above").

    (i) $X \setminus \overline{A} \;\Box\; \text{Int}(X \setminus A)$

    (ii) $X \setminus \text{Int}(A) \;\Box\; \overline{X \setminus A}$

    (iii) $\text{Int}(A \cup B) \;\Box\; \text{Int}(A) \cup \text{Int}(B)$

    (iv) $\text{Int}(A \cap B) \;\Box\; \text{Int}(A) \cap \text{Int}(B)$

    (v) $\overline{A \cup B} \;\Box\; \overline{A} \cup \overline{B}$

    (vi) $\overline{A \cap B} \;\Box\; \overline{A} \cap \overline{B}$

    (b) Give a complete proof of your solution to part (a) (v).

    (c) Give a counterexample to each instance in part (a) where equality fails.

    (d) Prove or give a counterexample: If $A \subseteq X$, then $\partial A = \overline{A} \setminus A$.

    (e) Prove or give a counterexample: If $U \subseteq X$ is open, then $U = \text{Int}(\overline{U})$.

    (f) Let X and Y be topological spaces. Prove or disprove: if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$. Determine in general the relationship between $\overline{A} \times \overline{B}$ and $\overline{A \times B}$.

2. (a) Prove the following.

    Proposition (Equivalent definition of continuity). A function $f : X \to Y$ of topological spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset $A \subseteq X$.

    (b) Let $f : X \to Y$ be a continuous function of topological spaces. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements in X converging to $a_\infty \in X$. Show that $\lim_{n \to \infty} f(a_n) = f(a_\infty)$.

3. Prove the following.

   Proposition (Equivalent definition of Hausdorff). A topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a closed subset of the product $X \times X$ with the product topology.

4. (a) Show that the continuous image of a Hausdorff topological space need not be Hausdorff. Specifically, find a Hausdorff topological space X, and topological space Y that is not Hausdorff, and a continuous surjective map $f : X \to Y$.

   (b) Let X and Y be topological spaces, and assume that Y is Hausdorff. Show that, if there exists a continuous injective map $f : X \to Y$, then X must also be Hausdorff.

   (c) Let X and Y be topological spaces, and assume that Y is Hausdorff. Let $A \subseteq X$. Show that the values of a continuous function $X \to Y$ on $\overline{A}$ are completely determined by its value on A. Specifically, let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Suppose that $A \subseteq X$ is a subset such that

   $$f(a) = g(a) \qquad \text{for all } a \in A.$$

   Prove that

   $$f(x) = g(x) \qquad \text{for all } x \in \overline{A}.$$

5. **(The Zariski Topology).** Fix $n \in \mathbb{N}$, and let $\mathbb{C}[x_1, \ldots, x_n]$ denote the set of polynomials in the variables $x_1, \ldots, x_n$ with complex coefficients. For a subset of polynomials $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we denote its *vanishing set*

   $$V(S) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid f(z_1, \ldots, z_n) = 0 \quad \text{for all } f \in S\} \subseteq \mathbb{C}^n.$$

   In this problem, we will see that there is a topology on $\mathbb{C}^n$, called the *Zariski topology*, whose closed sets are exactly the sets $V(S)$ for $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$. This topology is a foundational construction in algebraic geometry.

   (a) Let $\{S_i \mid i \in I\}$ be a collection of subsets of $\mathbb{C}[x_1, \ldots, x_n]$. Show that $V\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} V(S_i)$.

   (b) If $S_1, S_2 \subseteq \mathbb{C}[x_1, \ldots, x_n]$, prove that $V(S_1 \cdot S_2) = V(S_1) \cup V(S_2)$, where

   $$S_1 \cdot S_2 = \{f \cdot g \mid f \in S_1,\ g \in S_2\}.$$

   (c) Prove that the Zariski topology $\mathcal{T}_Z = \{U \subseteq \mathbb{C}^n \mid \mathbb{C}^n \setminus U = V(S) \text{ for some } S \subseteq \mathbb{C}[x_1, \ldots, x_n]\}$ is in fact a topology on $\mathbb{C}^n$.

   (d) Determine whether points $\{(z_1, \ldots, z_n)\} \subseteq \mathbb{C}^n$ are closed in the Zariski topology.

   (e) Show that the subspace topology on the subset $\mathbb{C} \cong \{(z, 0, \cdots, 0) \mid z \in \mathbb{C}\} \subseteq \mathbb{C}^n$ coincides with the Zariski topology on $\mathbb{C}$ (as defined by the polynomial ring $\mathbb{C}[x_1]$).

   (f) Determine whether the Zariski topology on $\mathbb{C}^n$ is Hausdorff. *Hint:* Use part (e).

6. **Bonus (Optional).** Let X be a topological space. Consider the two functions *closure* and *complement*

   $$\mathrm{Cl} : A \longmapsto \overline{A} \qquad \text{and} \qquad \mathrm{Co} : A \longmapsto X \setminus A$$

   on the power set of X.

   (a) Show that, starting with a fixed set $A \subseteq X$, you can form at most 14 sets by applying these two operations successively.

   (b) Find a subset $A \subseteq \mathbb{R}$ (with the standard topology) that achieves this maximum of 14 sets.