Recommended reading: Munkres Section 17, 18, 19.

Roughly similar content: Ch. 4.2 & 9.2, "Topology without tears" `http://www.topologywithouttears.net/topbook.pdf`

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Which of the following topologies on $\mathbb{R}$ are Hausdorff? Which satisfy the T_1 axiom (Assignment problem # 1)?

   - the standard topology
   - the discrete topology
   - the indiscrete topology
   - $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - topology generated by $\{[a, b) \mid a, b \in \mathbb{R}\}$
   - $\{U \mid 1 \in U\} \cup \{\varnothing\}$
   - $\{U \mid 1 \notin U\} \cup \{\mathbb{R}\}$
   - the cofinite topology
   - the cocountable topology

2. (a) Give an example of a continuous, invertible map $f : X \to Y$ of topological spaces whose inverse f^{-1} is not continuous.

   (b) Let $f : X \to Y$ be a bijection. Show that f^{-1} is continuous if and only if f is open.

3. Let $f : X \to Y$ be a homeomorphism of topological space $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Show that the map $U \longmapsto f(U)$ is a well-defined bijection between the open sets $\mathcal{T}_X$ in X and the open sets $\mathcal{T}_Y$ in Y.

4. Let $f : X \to Y$ be a function of topological spaces.

   (a) Let $A \subseteq X$ be a subset. Show that, if f is continuous, then the restriction of f to A

   $$\begin{aligned} f|_A : A &\longrightarrow Y \\ x &\longmapsto f(x) \end{aligned}$$

   is continuous (with respect to the topology on Y and the subspace topology on A.)

   (b) Suppose that Z is a subset of Y containing $f(X)$. Show that f is continuous if and only if the function

   $$\begin{aligned} X &\longrightarrow Z \\ x &\longmapsto f(x) \end{aligned}$$

   is continuous (with respect to the topology on X and the subspace topology on Z).

5. (a) Let $f : X \to Y$ be a homeomorphism. Show that $f^{-1} : Y \to X$ is also a homeomorphism.

   (b) Let $f : X \to Y$ and $g : Y \to Z$ be homeomorphisms. Show that $g \circ f : X \to Z$ is a homeomorphism.

   (c) Consider any set of topological spaces. Show that the relation

   $$X \sim Y \qquad \Longleftrightarrow \qquad X \text{ is homeomorphic to } Y$$

   is an equivalence relation on this set.

6. Let X and Y be homeomorphic topological spaces.

   (a) Show that X has the discrete topology if and only if Y does.

   (b) Show that X has the indiscrete topology if and only if Y does.

   (c) Show that X is Hausdorff if and only if Y is.

   (d) Show that X is a T_1-space if and only if Y is.

7. Let X_1, X_2, and X_3 be topological spaces. Show that there are homeomorphisms
$$\Big((X_1 \times X_2) \times X_3\Big) \cong \Big(X_1 \times (X_2 \times X_3)\Big) \cong (X_1 \times X_2 \times X_3).$$

8. (a) Let X be a topological space, and let $A \subseteq X$. View A as a topological space with the subspace topology. Show that the inclusion $A \to X$ is an embedding.
   (b) Let X, Y be topological spaces. Fix $x_0 \in X$ and fix $y_0 \in Y$. Show that the following maps are embeddings.
$$\begin{aligned} X &\longrightarrow X \times Y \\ x &\longmapsto (x, y_0) \end{aligned} \qquad\qquad \begin{aligned} Y &\longrightarrow X \times Y \\ y &\longmapsto (x_0, y) \end{aligned}$$

9. Determine whether the following functions are continuous, and whether they are homeomorphisms. All sets are subspaces of $\mathbb{R}$ with the standard topology.

   (a) $\begin{aligned} f : (-1,1) &\longrightarrow [0,1) \\ x &\longmapsto |x| \end{aligned}$ (b) $\begin{aligned} f : (-1,1) &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{x}{1-x^2} \end{aligned}$ (c) $\begin{aligned} f : \mathbb{R} &\longrightarrow [-1,1] \\ x &\longmapsto \sin(x) \end{aligned}$ (d) $\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^3 \end{aligned}$

10. (a) Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Verify that the box topology on $\prod_{i \in I} X_i$ is a topology.
   (b) Verify that the box topology is finer than (or equal to) the product topology.
   (c) Verify that the box and product topology are the same when I is finite.

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. **Definition (The T_1 Axiom).** A topological space X is called a *T_1-space* if, for every pair of distinct points $x, y \in X$, there is a neighbourhood U_x of x and a neighbourhood U_y of y such that $y \notin U_x$ and $x \notin U_y$. We call this condition the *T_1 axiom.*

   Note that the T_1 axiom is weaker than the Hausdorff condition, since we do not require the neighbourhoods U_x and U_y to be disjoint.

   (a) Give an example (with proof) of a topological space that is a T_1-space but is not Hausdorff.
   (b) Prove the following proposition.

   Proposition (Equivalent statement of the T_1 Axiom). Let X be a topological space. Show that X is a T_1-space if and only if the singleton set $\{x\}$ is closed for every $x \in X$.

   (c) Let X be a **finite** set, and suppose that $\mathcal{T}$ is a topology on X satisfying the T_1 axiom. Show that $\mathcal{T}$ is the discrete topology.
   (d) Prove the following proposition.

   Proposition (Limit points in T_1-spaces). Let X be a T_1-space, and let $A \subseteq X$. Then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.

   (e) Let X be a T_1-space, and let $A \subseteq X$. Show that the set A' of limit points of A is closed.
   (f) Show by example that, for a subset A of a general topological space, A' need not be closed.

2. Let $f : X \to Y$ be a function of topological spaces.
   (a) Suppose that X can be written as a union of **open** subsets $X = \bigcup_{i \in I} U_i$. Suppose moreover that for each $i \in I$, the restriction $f|_{U_i} : U_i \to Y$ of f to U_i is continuous with respect to the subspace topology on U_i. Show that f is continuous.

(b) Suppose that $X = A \cup B$ for **closed** sets A and B. Show that, if $f|_A : A \to Y$ and $f|_B : B \to Y$ are both continuous, then f is continuous.

(c) Let Y be an ordered set with the order topology. Let $f, g : X \to Y$ be continuous functions. Show that the "minimum" function $m(x)$ is continuous:

$$\begin{aligned} m &: X \to Y \\ m(x) &= \min\{f(x), g(x)\}. \end{aligned}$$

3. We saw on Quiz #2 that a function $Z \to X \times Y$ is continuous if and only if its coordinate functions are continuous. In this question, we will consider continuity of functions $X \times Y \to Z$.

Definition (Continuity in each variable). Let X, Y, Z be topological spaces, and $X \times Y$ the topological space with the product topology. Let $F : X \times Y \to Z$ be a function. Then F is *continuous in each variable separately* if for each $y_0 \in Y$, and for each $x_0 \in X$, the following maps are continuous.

$$\begin{aligned} X &\longrightarrow Z \\ x &\longmapsto F(x, y_0) \end{aligned} \qquad\qquad \begin{aligned} Y &\longrightarrow Z \\ y &\longmapsto F(y, x_0). \end{aligned}$$

(a) Show that, if F is continuous, then it is continuous in each variable.

(b) Show that the converse is false. *Hint:* Consider the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$F(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

You can use the following result from real analysis without proof. Here $\mathbb{R}$ has the standard topology.

Lemma. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function, and fix a point (x_0, y_0) in $\mathbb{R} \times \mathbb{R}$. If F is continuous at (x_0, y_0), then for any parameterized line

$$x(t) = x_0 + at, \qquad y(t) = y_0 + bt \qquad (a, b \in \mathbb{R} \text{ any constants}),$$

the limit $\lim_{t \to 0} F\big(x(t), y(t)\big)$ exists and equals $F(x_0, y_0)$.

4. Let $f : X \to Y$ be a contiuous function of topological spaces. Recall that the *graph* G of f is the subset

$$G = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y.$$

Then G is a topological space as a subspace of the space $X \times Y$ with the product topology. Show that X is homeomorphic to G.

5. **Product and box topologies.** Let $\{X_i\}_{i \in I}$ be a collection of topological spaces indexed by a set I.

(a) Verify that the product topology on $\prod_{i \in I} X_i$ is in fact a topology.

(b) Let $\pi_j : \prod_{i \in I} X_i \to X_j$ denote the projection onto the j^{th} coordinate. Show that this map is continuous with respect to the product topology, and moreover that the product topology is the coarsest topology on $\prod_{i \in I} X_i$ making the projection maps π_j continuous for all $j \in I$.

(c) Let $f : X \to \prod_{i \in I} X_i$ be given by the equation $f(x) = \Big(f_i(x)\Big)_{i \in I}$ with functions $f_i : X \to X_i$. Show that f is continuous (with respect to the product topology) if and only if each coordinate function f_i is continuous.

6. **Bonus (Optional).** Prove that the interval $I = (0, 1) \subseteq \mathbb{R}$ is not homeomorphic to the circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2.$$