Recommended reading: Munkres Section 19, 20

Roughly similar content: Ch. 9.2, 6,1 "Topology without tears" `http://www.topologywithouttears.net/topbook.pdf`

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Carefully explain why the following map is continuous and injective, but not an embedding.

$$\begin{aligned} f : [0,1) &\longrightarrow \mathbb{R} \\ x &\longmapsto (\cos(2\pi x), \sin(2\pi x)). \end{aligned}$$

2. Show that all parts of Assignment Problem #1 also hold when the product topology on $\prod_{i \in I} X_i$ and its subspaces is replaced by the box topology on $\prod_{i \in I} X_i$ and its subspaces.

3. Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be families of nonempty topological spaces. Assume for each i that $f_i : X_i \to Y_i$ is a function. Let $\prod f_i$ denote the function

$$\begin{aligned} \prod f_i : \prod_{i \in I} X_i &\longmapsto \prod_{i \in I} Y_i \\ (x_i)_{i \in I} &\longmapsto \big(f_i(x_i)\big)_{i \in I} \end{aligned}$$

Show that the following hold when $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$ are both given the product topology, and when they are both given the box topology.

(a) $\prod f_i$ is continuous if and only if f_i is continuous for each i.

(b) $\prod f_i$ is injective if and only if f_i is injective for each i.

(c) $\prod f_i$ is surjective if and only if f_i is surjective for each i.

(d) $\prod f_i$ is bijective if and only if f_i is bijective for each i.

(e) $\prod f_i$ is open if and only if f_i is open for each i.

(f) $\prod f_i$ is a homeomorphism if and only if f_i is a homeomorphism for each i.

4. Let $X = \{a, b, c\}$. Which of the following functions define a metric on X?

(a)

$$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 3 \end{aligned}$$

(b)

$$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 4 \end{aligned}$$

5. (a) Weproved that, for continuous funcftions from $\mathbb{R} \to \mathbb{R}$, that the open-set definition of continuity is equivalent to the ϵ-δ definition. Verify that these definitions are equivalent for functions of arbitrary metric space $X \to Y$.

(b) You proved that, for sequences in $\mathbb{R}$, that the open-set definition of convergence is equivalent to the ϵ-N definition of convergence. Verify that these definitions are equivalent for sequences in an arbitrary metric space X.

6. Let (X, d) be a metric space. Show that the set $\{B_r(x) \mid x \in X, r \in \mathbb{Q}\}$ of balls with **rational** radius generate the topology induced by d.

7. Let (X, d) be a metric space.
   (a) Is $\varnothing$ a bounded set?
   (b) Show that any finite subset of a metric space is bounded.

8. Let (X, d) be a metric space, and $A \subseteq X$. Show that the following conditions are equivalent (and so can each be taken as the definition of *boundedness* for A.)
   (i) There is some $M \in \mathbb{R}$ such that $d(x, y) < M$ for every $x, y \in A$.
   (ii) There is some $a \in A$ and $R \in \mathbb{R}$ such that $A \subseteq B_R(a)$.
   (iii) For every $a \in A$, there is some $R_a \in \mathbb{R}$ such that $A \subseteq B_{R_a}(a)$.

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. **(The product topology).** Let $\{X_i\}_{i\in I}$ be a family of topological spaces.
   (a) For each i, let $A_i \subseteq X_i$. Consider the following two ways that we can topologize the product $A = \prod_{i\in I} A_i$. The first is to take the subspace topology on A_i for each i, and then the product topology on A. The second is to view A as subspace of the product $\prod_{i\in I} X_i$ with the product topology. Prove that these two topologies on A are equal.
   (b) Suppose that X_i is Hausdorff for all i. Prove that the both the box topology and product topology on the product $\prod_{i\in I} X_i$ are Hausdorff.
   (c) For each i, let $A_i \subseteq X_i$. Consider $\prod_{i\in I} X_i$ with the product topology. Show that $\prod_{i\in I} \overline{A_i} = \overline{\prod_{i\in I} A_i}$.

   Remark: The analogous statements hold for the box topology on $\prod_{i\in I} X_i$.

2. Let $\mathbb{R}^\omega = \prod_{\mathbb{N}} \mathbb{R}$ (so an element of $\mathbb{R}^\omega$ is precisely a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers.)
   (a) Let $X \subseteq \mathbb{R}^\omega$ be the subset of all sequences $(a_n)_{n\in\mathbb{N}}$ that are eventually zero, that is,
   $$X = \{(a_n)_{n\in\mathbb{N}} \mid \ a_n = 0 \text{ for all but finitely many values of } n \ \}.$$
   Find the closure of X in the box topology, and in the product topology, on $\mathbb{R}^\omega$.
   (b) Determine whether the following function is continuous in with respect to the box topology, and with respect to the product topology.
   $$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^\omega \\ t &\longmapsto (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots,) \end{aligned}$$
   (c) State all limits of the following sequences with respect to the box topology, and with respect to the product topology. **No justification needed.**

   (i)
   $$\begin{aligned} \mathbf{a_1} &= (1, 0, 0, 0, \ldots) \\ \mathbf{a_2} &= (\tfrac{1}{2}, 0, 0, 0, \ldots) \\ \mathbf{a_3} &= (\tfrac{1}{3}, 0, 0, 0, \ldots) \\ \mathbf{a_4} &= (\tfrac{1}{4}, 0, 0, 0, \ldots) \\ &\vdots \end{aligned}$$

   (ii)
   $$\begin{aligned} \mathbf{b_1} &= (1, 0, 0, 0, \ldots) \\ \mathbf{b_2} &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots) \\ \mathbf{b_3} &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots) \\ \mathbf{b_4} &= (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0 \ldots) \\ &\vdots \end{aligned}$$

3. (a) Let (X, d) be a metric space. Prove that the basis for the metric topology $\mathcal{B} = \{B_r(x) \mid x \in X, r > 0\}$ is in fact a basis.

(b) Let (X, d) be a metric space. Prove that $U \subseteq X$ is open in the metric topology if and only if, for all $x \in U$, there is some $\epsilon > 0$ so that $B_\epsilon(x) \subseteq U$.

(c) Determine whether the following functions define metrics on the corresponding sets.

(i) Let $X = \mathbb{R}$. Define

$$\begin{aligned} d : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ d(x, y) &= (x - y)^2. \end{aligned}$$

(ii) Let X be any set. Define

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ d(x, y) &= \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases} \end{aligned}$$

(d) **Definition (Bounded subset of a metric space).** Let (X, d) be a metric space, and $A \subseteq X$. Then A is *bounded* if there exists some $M \in \mathbb{R}$ such that $d(x, y) < M$ for all $x, y \in A$.

Suppose that (X, d) is a metric space. Show that there exists some metric $\tilde{d}$ on X that induces the same topology on X as d, (ie, d and $\tilde{d}$ are equivalent), but such that X is bounded with respect to the metric $\tilde{d}$.

4. (a) Let (X, d) be a metric space. Prove that a subset $C \subseteq X$ is closed if and only if it satisfies the following property: given any convergent sequence $(a_n)_{n \in \mathbb{N}}$ of points in C, its limit a_∞ is contained in C.

(b) Show that the above equivalence does not hold for subsets of arbitrary topological spaces.

(c) **Definition (Sequential continuity).** A function $f : X \to Y$ of topological spaces is *sequentially continuous* if, for every convergent sequence $(a_n)_{n \in \mathbb{N}}$ in X, then the sequence $\big(f(a_n)\big)_{n \in \mathbb{N}}$ in Y converges, and

$$\lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right).$$

Prove the following result.

Proposition. Let $f : X \to Y$ be a function of topological spaces, and assume that X is metrizable. Show that f is continuous if and only if it is sequentially continuous.

5. Fix integers $n, p \geq 1$. Consider the following function

$$\begin{aligned} d_p : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (\mathbf{x}, \mathbf{y}) &\longmapsto \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}. \end{aligned}$$

It is a fact that you do not need to prove (but which follows from *Minkowski's inequality*, `https://en.wikipedia.org/wiki/Minkowski_inequality`) that d_p is a metric on $\mathbb{R}^n$.
Further define the function

$$\begin{aligned} d_\infty : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (\mathbf{x}, \mathbf{y}) &\longmapsto \max_{i=1,\ldots,n} |x_i - y_i|. \end{aligned}$$

(a) Show that d_∞ is a metric.

(b) Sketch the unit balls $B_1(\mathbf{0})$ in $(\mathbb{R}^2, d_p)$ for $p = 1, 2, \infty$.

(c) Show that metrics d_p are equivalent metrics on $\mathbb{R}^n$ for all $p \in \mathbb{N} \cup \{\infty\}$. In particular, they are all equivalent to the standard Euclidean metric d_2.

(d) Show that these metrics are equivalent to the product topology on $\mathbb{R}^n$, as a product of copies of $\mathbb{R}$ with the usual topology.