Recommended reading: Munkres Section 20, 21, 22.

Roughly similar content: (Notes by Bob Gardner)
Metric Topology `http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-20.pdf`,
`http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-21.pdf`
Quotient Topology `https://faculty.etsu.edu/gardnerr/5210/notes/Munkres-22.pdf`

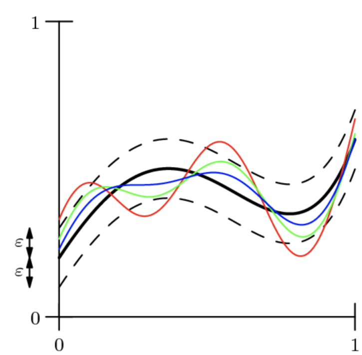
Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **(Real analysis review).**
   (a) Show that addition, subtraction, and multiplication are continuous functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (with the standard topologies). Show that the quotient operation is a continuous function $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$.
   (b) Let $f, g : X \to \mathbb{R}$ be continuous functions. Conclude that the functions $(f+g)$, $(f-g)$ and $(f \cdot g)$ are continuous. If $g(x) \neq 0$ for all $x \in X$, conclude that the function f/g is continuous.

2. Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. Show that the restriction $d|_{Y \times Y}$ of d to $Y \times Y \subseteq X \times X$ defines a metric on Y. Conclude that any subset of a metric space inherits a metric space structure.

3. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions from a topological space X to a metric space Y. See Assignment Problem # 3 for the definitions of pointwise and uniform convergence.
   (a) Show that uniform convergence implies pointwise convergence.
   (b) Use the following picture of functions f_1, f_2, f_3, and f_∞ to explain the concept of uniform convergence of functions $\mathbb{R} \to \mathbb{R}$, and how it differs from pointwise convergence.



4. (a) Let X be a set. Recall that the *discrete metric* on X is the metric $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ for all $x, y \in X$. Show that this metric induces the discrete topology on X.
   (b) Consider $\mathbb{N} \subseteq \mathbb{R}$. Show that the usual Euclidean metric on $\mathbb{N}$ induces the discrete topology.
   (c) Consider $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Show that the Euclidean metric on A induces the discrete topology.
   (d) For a metric space (X, d), consider the set $S_d = \{d(x,y) \mid x, y \in X\}$. Conclude that it is possible to have two equivalent metrics d and $\tilde{d}$ where S_d is bounded above and $S_{\tilde{d}}$ is not; and similiary it is possible to have two equivalent metrics d and $\tilde{d}$ where S_d is bounded below and $S_{\tilde{d}}$ is not.

5. In class we proved that the box topology on $\mathbb{R}^\omega$ is not metrizable. Explain where our proof would fail for the product topology on $\mathbb{R}^\omega$ (which is, in fact, metrizable).

6. Let X be a topological space, A a set, and $p : X \to A$ a surjective map. Recall that we defined the *quotient topology* on A to be the $\mathcal{T} = \{U \subseteq A \mid p^{-1}(A) \text{ is open in } X\}$. Verify that $\mathcal{T}$ is in fact a topology.

7. Verify that the composite of quotient maps is a quotient map.

8. Show that any quotient space of a discrete topological space has the discrete topology, and that any quotient space of an indiscrete topological space has the indiscrete topology.

9. (a) Let $f : X \to Y$ be a continuous surjective map of topological spaces. Show that if f is either open or closed, then f is a quotient map. We will see in Assignment Problem #4 part (b) that the converse statements do not hold.

   (b) Give an example of a map $X \to Y$ of topological spaces that is open but not closed, and an example of map that is closed but not open.

10. Consider the following functions from $\mathbb{R}$ to the set $X = \{a, b, c\}$. For each function, find the quotient topology on X induced by the function.

$$f_1 : \mathbb{R} \to \{a, b, c\} \qquad f_2 : \mathbb{R} \to \{a, b, c\} \qquad f_3 : \mathbb{R} \to \{a, b, c\}$$

$$f_1(x) = \begin{cases} a, \ x \in (-\infty, 0) \\ b, \ x = 0, \\ c, \ x \in (0, \infty) \end{cases} \qquad f_2(x) = \begin{cases} a, \ x = 0 \\ b, \ x = 1 \\ c, \ x \neq 0, 1 \end{cases} \qquad f_3(x) = \begin{cases} a, \ x \in (-\infty, 0) \\ b, \ x = [0, 1) \\ c, \ x \in [1, \infty) \end{cases}$$

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. Let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence of points in the product space $\prod_{i\in I} X_i$.

   (a) Show that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to a point $\mathbf{x}_\infty \in \prod_{i\in I} X_i$ in the product topology if and only if, for every $i \in I$, the sequence $(\pi_i(\mathbf{x}_n))_{n\in\mathbb{N}}$ in X_i converges to the point $\pi_i(\mathbf{x}_\infty)$ in X_i.

   (b) Prove or disprove the same statement for the box topology on $\prod_{i\in I} X_i$.

2. Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. In Warm-Up Problem #2 (which you should check but do not need to write up), we saw that the restriction $d|_{Y\times Y}$ of d to points in Y defines a metric on Y. There are now two ways we can define a topology on Y: as the topology induced by the restriction of the metric d to Y, or as a subspace of X with the topology induced by d. Verify that these two topologies are equal.

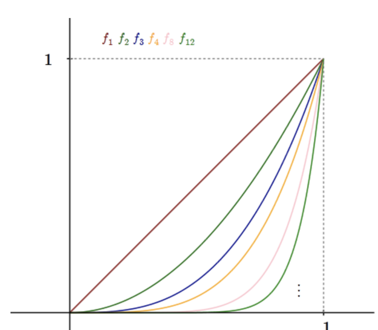
3. **Definition (Pointwise and Uniform Convergence).** Let X be a topological space, and (Y, d) a metric space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : X \to Y$.

   - Then the sequence $(f_n)_{n\in\mathbb{N}}$ *converges at a point* $x \in X$ if the sequence $(f_n(x))_{n\in\mathbb{N}}$ of points in Y converges.
   - The sequence $(f_n)_{n\in\mathbb{N}}$ *converges pointwise* to a function $f_\infty : X \to Y$ if for every point $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ of points in Y converges to the point $f_\infty(x) \in Y$.
   - The sequence $(f_n)_{n\in\mathbb{N}}$ *converges uniformly* to a function $f_\infty : X \to Y$ if for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ so that $d\big(f_n(x), f_\infty(x)\big) < \epsilon$ for every $n \geq N$ and $x \in X$.

   In other words, if the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f_∞, then for each $\epsilon > 0$ the choice of N may depend on the point $x \in X$. To converge uniformly to f_∞, there must exist a choice of N that is independent of the point x.

   (a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of **continuous** functions from a topological space X to a metric space Y. Suppose that this sequence converges uniformly to a function $f_\infty : X \to Y$. Show that f_∞ is continuous.

   (b) Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $[0, 1]$ to $[0, 1]$ (with the usual metric) defined by $f_n(x) = x^n$. Show that this sequence converges pointwise, but conclude from part (a) that it does not converge uniformly.



$f_n(x) = x^n$

(c) **Definition (The uniform metric on $\mathbb{R}^I$).** Given an index set I, define the *uniform metric* on $\mathbb{R}^I$ as follows. For points $\mathbf{a} = (a_i)_{i\in I}$ and $\mathbf{b} = (b_i)_{i\in I}$, let

$$d(\mathbf{a}, \mathbf{b}) = \sup_{i\in I} \Big\{ \min\{|a_i - b_i|, 1\} \Big\}.$$

(You do not need to verify that this is a metric). The topology induced by d is called the *uniform topology.*

Prove that the uniform topology is finer than (or equal to) the product topology, and coarser than (or equal to) the box topology on $\mathbb{R}^I$.

(d) Consider a sequence $f_n : X \to \mathbb{R}$ of functions from a topological space X to $\mathbb{R}$. We can view these functions as elements of the product space $\mathbb{R}^X$. Prove that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly if and only if it converges as a sequence of elements in $\mathbb{R}^X$ with the uniform topology.

4. (a) Let X be a topological space. Let $p : X \to Y$ be a surjective map to a set Y, and endow Y with the quotient topology induced by p. Show that $C \subseteq Y$ is closed in Y if and only if $p^{-1}(C)$ is closed in X.

(b) Let $\pi_1 : \mathbb{R} \times \mathbb{R}$ be projection onto the first factor. Consider the resutriction $\pi_1|_A$ of π_1 onto the subspace $A = \{(x, y) \mid x \geq 0 \text{ and/or } y = 0\}$. Show that $\pi_1|_A$ is a quotient map, but that it is neither open nor closed.

(c) Let $p : X \to Y$ be a quotient map of topological spaces. Show by example that the restriction of p to a subspace $A \subseteq X$ need not give a quotient map from A to $p(A)$.

(d) Prove the following result.

Theorem (Restrictions of quotient maps). Let $p : X \to Y$ be a quotient map of topological spaces. If $A \subseteq X$ is an open subset that is saturated with respect to p, then the restriction of p to A defines a quotient map from A to $p(A)$.

Remark: A similar argument shows that, if A is closed, then $p|_A$ is a quotient map.

5. Consider the topology on $\mathbb{R}$ generated by the basis

$$\{(a, b) \mid a, b \in \mathbb{R}\} \cup \{(a, b) \setminus K \mid a, b \in \mathbb{R}\}, \qquad K = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}.$$

This is called the *K-topology* on $\mathbb{R}$. Let Y be the quotient space obtained from $\mathbb{R}$ by identifying all elements of K to a single point, and let $p : \mathbb{R} \to Y$ be the quotient map.

(a) Show that Y is a T_1-space, but isn't Hausdorff. Conclude in particular from Homework 5 Problem #3 that the diagonal is not closed in $Y \times Y$.

(b) Show that the product map $\mathbb{R} \times \mathbb{R} \to Y \times Y$ is not a quotient map.

This exercise shows that the product of quotient maps need not be a quotient map.

6. **Bonus (Optional).**

Definition (Topological group). A *topological group* is a topological space that is also a group, such that the mutliplication and inversion maps are continuous:

$$G \times G \to G \qquad\qquad G \to G$$
$$(g, h) \mapsto gh \qquad\qquad g \mapsto g^{-1}.$$

(a) Prove that any open subgroup H of G is also closed.

(b) Suppose $H \subseteq G$ is a subgroup. Show that its topological closure $\overline{H}$ is also a subgroup.

(c) Suppose that $H \subseteq G$ is a normal subgroup. Show that the quotient topology on the quotient group G/H makes it a topological group.

(d) Show that G/H is Hausdorff if and only if H is closed.