Recommended reading: Munkres Section 23.

Roughly similar content: Hatcher Chapter 2 `https://pi.math.cornell.edu/~hatcher/Top/TopNotes.pdf`

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Show that a space X is connected if and only if the only subsets of X that are both open and closed are X and $\varnothing$.
2. Consider $\{0,1\}$ as a topological space with the discrete topology. Show that a topological space $(X, \mathcal{T})$ is disconnected if and only if there is a continuous **surjective** function $X \to \{0,1\}$.
3. Show that connectedness is a topological property. In other words, show that, if X is connected, then so is any space that is homeomorphic to X.
4. Show that the following topological spaces (with the Euclidean metric) are disconnected.

   (a) $\mathbb{N}$ (b) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ (c) $(0,1) \cup \{5\}$ (d) $(0,1) \cup (1,3)$ (e) $\mathbb{R} \setminus \mathbb{Q}$

5. Show that singleton sets $\{x\}$ in a topological space are always connected.
6. Suppose that X is connected topological space. Explain why X is also connected with respect to any coarser topology. What about finer topologies?
7. Let $A \subseteq X$ be a subspace. Show that A is not connected if and only if the following condition holds: there exist open subsets U, V in X such that $U \cap A$ and $V \cap A$ are nonempty, and $A \subseteq U \cup V$.
8. (a) Show by example that disconnected spaces can have connected subspaces.

   (b) Show by example that connected spaces can have disconnected subspaces.
9. Let $X = \{a, b, c, d\}$ with the topology
$$\Big\{\varnothing, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}\Big\}.$$
   Find a nonempty subset of X that is connected, and a nonempty subset of X that is disconnected.

Assignment questions

(Hand these questions in! Unless otherwise indicated, give a complete, rigorous justification for each solution.)

1. Let A be a connected subset of X, and suppose that $A \subseteq B \subseteq \overline{A}$. Show that B is connected.
   Remark: In particular, this shows that if A is connected, then $\overline{A}$ is connected.
2. Prove the following theorem.

   Theorem (Continuous images of connected sets). Suppose that $f : X \to Y$ is a continuous map of topological spaces. If X is connected, then $f(X)$ is connected.
3. **Definition (Connected components of a topological space).** Let $(X, \mathcal{T}_X)$ be a topological space. A subset $C \subseteq X$ is called a *connected component* of X if

   (i) C is connected;

   (ii) if C is contained in a connected subset A, then $C = A$.

(a) Let $x \in X$. Show that the set

$$\bigcup_{\substack{A \text{ is a connected set,} \\ x \in A}} A$$

is a connected component of X.

(b) Show that X is the disjoint union of its connected components. In other words, show that the decomposition of X into connected components is a partition of X.
Remark: We can therefore redefine *connected components* as the equivalence classes defined by the following equivalence relation on points of X:

$$x \sim y \qquad \Longleftrightarrow \qquad \text{there exists a connected subset of } X \text{ containing } x \text{ and } y.$$

(c) Show that any connected component of X is closed. (*Hint*: Problem # 1).

(d) Determine the connected components of $\mathbb{Q}$ (with the standard topology). Deduce that connected components need not be open.