1. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, write "`False`". **No justification necessary.**

(i) Let S be the set of sequence of rational numbers that are eventually zero. Then S is countable.

(ii) Let S be the set of sequences of rational numbers that converge to zero. Then S is countable.

(iii) Given a countable collection of uncountable sets $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$, their intersection must be uncountable.

(iv) Let X be a space and $A \subseteq X$. If A is closed in X, then A will also be closed with respect to any finer topology on X.

(v) Let X be a space and $A \subseteq X$. If A is closed in X, then A will also be closed with respect to any coarser topology on X.

(vi) Let X be a set. If $\mathcal{T}_1$ is a finer topology on X than $\mathcal{T}_2$, then the identity map on X is necessarily continuous when viewed as function from $(X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$.

(vii) If X is a Hausdorff space, then limits of sequences in X are unique.

(viii) If X is a space where limits of sequences are unique, then X is Hausdorff.

(ix) Let X be a totally ordered set with the order topology, and let $a, b \in X$. Then $[a, b]$ is the closure of (a, b).

(x) Let $f : X \to Y$ be a homeomorphism, and let $A \subseteq X$. Then f restricts to a homeomorphism $f|_A : A \to f(A)$ between the subspaces A and $f(A)$.

(xi) Two sets X and Y with the discrete topology are homeomorphic if and only if they have the same cardinality.

(xii) Two sets X and Y with the cofinite topology are homeomorphic if and only if they have the same cardinality.

(xiii) Let $f : X \to Y$ be a continuous injective map. If f is an open map, then f is an embedding.

(xiv) Let $f : X \to Y$ be a continuous injective map. If f is a closed map, then f is an embedding.

(xv) If the closure of a subspace A of a space X is connected, then $\text{Int}(A)$ is connected.

(xvi) If A is a subspace of X such that $\text{Int}(A)$ is connected, then A is connected.

(xvii) Let $f : (0, 1) \to \mathbb{R}$ be the function $f(x) = \sin\left(\frac{1}{x}\right)$. Then there is no way to define f at 0 to extend f to a continuous function $f : [0, 1) \to \mathbb{R}$.

(xviii) If X is a connected topological space, then X is also connected with respect to any coarser topology.

(xix) The space $\mathbb{R}^\omega$ in the product topology is connected.

(xx) An uncountable set with the cofinite topology is second countable.

(xxi) The space $\mathbb{N}$ with the cofinite topology is second countable.

(xxii) Let X be any set endowed with the cofinite topology. Then X is compact.

(xxiii) The space $\mathbb{R}^\omega$ in the product topology is normal.

(xxiv) The space $\mathbb{R}^\omega$ in the uniform topology is normal.

(xxv) Every metric space can be embedded isometrically into a complete metric space.

(xxvi) Euclidean space $\mathbb{R}^n$ is a Baire space.

(xxvii) Every totally bounded space is bounded.

(xxviii) Every compact metric space is complete.

(xxix) Every complete metric space is compact.

(xxx) There exists a continuous, surjective path $[0,1] \to [0,1]^2$.

(xxxi) Let X be a nonempty Baire space, and suppose that $X = \bigcup_{n\in\mathbb{N}} B_n$ for some countable collection of subsets B_n. Then $\overline{B_n}$ must have nonempty interior for some n.

(xxxii) The set of functions $\{f_n(x) = \frac{x}{n} \mid n \in \mathbb{N}\}$ in $\mathscr{C}(\mathbb{R},\mathbb{R})$ is equicontinuous.

(xxxiii) The set of functions $\{f_n(x) = x^n \mid n \in \mathbb{N}\}$ in $\mathscr{C}([0.1],\mathbb{R})$ is equicontinuous.

(xxxiv) A product of arbitrarily many compact spaces is compact in the product topology.

2. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, state a counterexample. **No justification necessary.**

   Note: You can get partial credit for correctly writing "`False`" without a counterexample.

(i) There does not exist a set whose power set is countably infinite.

(ii) Let S be a subspace of a topological space X, and let $A \subseteq X$. Then the closure of $A \cap S$ is in S is equal to $\overline{A} \cap S$, where $\overline{A}$ denotes the closure of A in X.

(iii) Let A be a connected subspace of a space X. Then $\overline{A}$ is connected.

(iv) Let X be a topological space, and $S \subseteq X$. If S is open, then $S \cap \partial S = \varnothing$.

(v) Let X be a topological space, and $S \subseteq X$. Then $\operatorname{Int}(S) = \operatorname{Int}(\overline{S})$.

(vi) Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then $\overline{X \setminus A} = X \setminus \overline{A}$.

(vii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X has the discrete topology, then so does the subspace $f(X) \subseteq Y$.

(viii) Every Hausdorff space is metrizable.

(ix) Let $\{A_i\}_{i\in I}$ be a collection of path-connected subspaces of a space X, such that $\bigcap_{i\in I} A_i \neq \varnothing$. Then $\bigcup_{i\in I} A_i$ is path-connected.

(x) If X is a metric space, then every compact subset of X is closed and bounded.

(xi) If X is a metric space, then every closed and bounded subset of X is compact.

(xii) If X is a limit point compact space, then X is also limit point compact with respect to any coarser topology.

(xiii) Let $f : X \to Y$ be a continuous function from a limit point compact space X to a space Y. Then $f(X)$ is limit point compact.

(xiv) A closed subset $A \subseteq X$ of a limit point compact space X is also limit point compact.

(xv) Any second countable space X is separable.

(xvi) Every metrizable space is second countable.

(xvii) Suppose that $(X, \mathcal{T})$ is a topological space that is first countable. Then any coarser topology on X will also be first countable.

(xviii) Suppose that $(X, \mathcal{T})$ is a topological space that is first countable. Then any finer topology on X will also be first countable.

(xix) A subspace of a second countable space is second countable.

(xx) A product of Lindelöf spaces is Lindelöf.

(xxi) The continuous image of a normal space is normal.

(xxii) A space X is locally compact and Hausdorff if and only if it is homeomorphic to an open subset of a compact Hausdorff space.

(xxiii) Let $f : X \to Y$ be a continuous map of metric spaces, and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then $\Big(f(x_n)\Big)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(xxiv) A subspace S of $\mathbb{R}^n$ with the Euclidean metric is complete if and only if it is closed.

(xxv) Every totally bounded metric space is compact.

3. Find the interior, closure, and boundary of each of the following subsets of $\mathbb{R}^\omega$, with respect to
   (*i*) the box topology, (*ii*) the uniform topology, (*iii*) the product topology.

   (a) the set of all bounded sequences

   (b) the set of all constant sequences

   (c) the set of all non-constant sequences

   (d) the set of all sequences that are eventually constant

   (e) the set of all sequences of rational numbers

   (f) the set of all sequences of natural numbers

   (g) the set of all sequences of strictly positive numbers

   (h) the set of all sequences of nonnegative numbers

4. Let (X, d) be a metric space, and let $A \subseteq X$. For $x \in X$, define

$$d(x, A) = \inf_{a\in A} d(x, a).$$

   (a) Prove that following the "distance to A" function is a continuous function on X,

$$d_A : X \to \mathbb{R}$$
$$d_A(x) = d(x, A).$$

   (b) Suppose that A is compact. Let $x \in X$. Show that there is some $a \in A$ so that $d(x, A) = d(x, a)$.

   (c) Let A be any subset of X. Show that $\overline{A} = \{x \mid d(x, A) = 0\}$.

5. (a) Let X and Y be spaces, and let X^* and Y^* be partitions of X and Y, respectively, with the quotient topology. Under what conditions will a continuous map $f : X \to Y$ induce a well-defined, continuous map $f^* : X^* \to Y^*$?

   (b) Let T denote the quotient of $\mathbb{R}$ by the equivalence relation $x \sim 2\pi x$; T is homeomorphic to the unit circle. Which continuous functions $f : \mathbb{R} \to \mathbb{R}$ descend to continuous functions $f^* : T \to T$?

6. Let X_i, $i \in I$ be a collection of topological spaces, and let $X = \prod_{i\in I} X_i$ denote their product in the box topology. Show that a sequence $((x_i^n)_{i\in I})_{n\in\mathbb{N}}$ of points in X converges if and only if it satisfies the following: Firstly, it converges pointwise at each index $i \in I$. In other words, for each $i \in I$ the sequence $(x_i^n)_{n\in\mathbb{N}}$ in X_i converges. Secondly, for all but finitely many i, the sequence $(x_i^n)_{n\in\mathbb{N}}$ must be eventually constant.

7. Suppose that there are embeddings $X \to Y$ and $Y \to X$. Show by example that this does not imply that X and Y are homeomorphic.

8. **Definition (Convex subsets of $\mathbb{R}^n$).** Let A be a subset of $\mathbb{R}^n$ (with the Euclidean metric). Then A is called *convex* if $t\mathbf{x} + (1-t)\mathbf{y} \in A$ for every $\mathbf{x}, \mathbf{y} \in A$ and any $t \in [0, 1]$.

   Prove that any convex subset of $\mathbb{R}^n$ is connected.

9. (a) Show that the unit interval $[0, 1]$ cannot be partitioned into a countable union of more than one closed sets. *Hint:*
      (i) Show that $[0, 1]$ cannot be partitioned into a finite number (of at least two) closed sets $C_1, C_2, \ldots, C_n$.
      (ii) Suppose that $[0, 1]$ were partitioned into a countably infinite number of closed sets $\{C_n\}_{n\in\mathbb{N}}$. Show that each C_n must have nonempty boundary.
      (iii) Let $B = \bigcup_{n\in\mathbb{N}} \partial C_n$. Show that, for each n, the boundary ∂C_n has empty interior as a subspace of B.
      (iv) Apply the Baire category theorem to reach a contradiction.

   (b) Consider the natural numbers $\mathbb{N}$ with the cofinite topology. Show that $\mathbb{N}$ is locally connected but not locally path-connected.

10. (a) Prove that any path-connected space is connected.
    (b) Prove that any locally path-connected space is locally connected.
    (c) State an example of a space that is not connected or path-connected
    (d) State an example of a space that is connected but not path-connected.
    (e) State an example of a space that is connected and path-connected but not locally connected or locally path-connected
    (f) State an example of a space that is connected and locally connected, but not path-connected or locally path-connected.
    (g) State an example of a space that is connected, path-connected, locally connected, but not locally path-connected.
    (h) State an example of a space that is locally connected, but not locally path-connected, connected, or path-connecterd.

11. Let $\mathbb{R}_K$ denote $\mathbb{R}$ with the K-topology. In this problem we will show that $\mathbb{R}_K$ is connected but not path-connected.
    (a) Explain why $(-\infty, 0)$ and $(0, \infty)$ inherits their usual topologies as subsaces of $\mathbb{R}_K$.
    (b) Show that $\mathbb{R}_K$ is connected.
    (c) Show that any interval containing $[0, 1] \subseteq \mathbb{R}_K$ is non-compact.

(d) Show that $\mathbb{R}_K$ is not path-connected.

12. (a) Let A and B be compact subspaces of a topological space X. Show that $A \cap B$ need not be compact. *Hint:* Consider $\mathbb{R}$ with the topology $\{U \mid 0, 1 \notin U\} \cup \{\mathbb{R}\}$.
    (b) Let A and B be compact subspaces of a Hausdorff space X. Show that $A \cap B$ is compact.

13. In this problem, we will show that **any** nonempty set admits a topology with respect to which it is compact and Hausdorff. Specifically, let X be a nonempty set, and let x_0 be a distinguished element of X. Let
$$\mathcal{T} = \{A \subseteq X \mid x_0 \notin A \text{ or } X \setminus A \text{ is finite }\}.$$
    (a) Show that $\mathcal{T}$ defines a topology on X.
    (b) Verify that $(X, \mathcal{T})$ is Hausdorff.
    (c) Verify that $(X, \mathcal{T})$ is compact.

14. Let $p : X \to Y$ be a closed, continuous, surjective map, with the property that $p^{-1}(y)$ is compact for all $y \in Y$. Suppose that Y is compact. Prove that X is compact.

15. Let X be compact and Y be Hausdorff. Then any continuous bijection $f : X \to Y$ is a homeomorphism.

16. Let $X \neq \varnothing$ be a compact Hausdorff space. Show that if X has no isolated points, then X is uncountable.

17. Show that a connected metric space with at least 2 points must have uncountably many points.

18. Let $X = [0, 1]^\omega$ in the uniform topology. Show that X is not limit point compact, and therefore not compact.

19. Determine where $\mathbb{R}^\omega$ with the box topology is first countable, or second countable.

20. Let $\mathbb{R}_\ell$ denote $\mathbb{R}$ with the *lower limit topology*, the topology generated by the basis $\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}\}$. This space is called the *Sorgenfrey line.*
    (a) Show that the lower limit topology is finer than the standard topology on $\mathbb{R}$.
    (b) Show that $\mathbb{R}_\ell$ is Hausdorff.
    (c) Show that $\mathbb{R}_\ell$ is first countable and separable.
    (d) Show that $\mathbb{R}_\ell$ is not second countable.
    (e) Show that $\mathbb{R}_\ell$ is not metrizable.
    (f) Show that $[0, 1] \subseteq \mathbb{R}_\ell$ is not limit point compact.

21. Let X be a separable metrizable space. Prove that any subspace $A \subseteq X$ is also separable.

22. Let X be a regular space. Show that every pair of points in X have neighbourhoods with disjoint closures.

23. Let $X = \prod_{i \in I} X_i$ be a product of nonempty spaces.
    (a) Suppose X is regular. Show that X_i is regular for all i.
    (b) Suppose X is normal. Show that X_i is normal for all i.

24. **Definition (Arens–Fort space).** Let $X = \mathbb{N} \times \mathbb{N}$ with the following topology,

$$\{\, U \mid (1,1) \notin U\} \cup \left\{ U \;\middle|\; \begin{array}{l} (1,1) \in U, \text{ and } U \text{ contains all but a finite} \\ \text{number of points in all but a finite number of colums} \end{array} \right\}.$$

This space is called *Arens–Fort space.*

Recall that a *column* of $\mathbb{N} \times \mathbb{N}$ is a subset of form $\{(m,n) \mid n \in \mathbb{N}\}$ for some fixed m.

(a) Show that X is Hausdorff.

(b) Show that X is regular.

(c) Show that X is not compact.

(d) Show that X is not first countable (and therefore not second countable or metrizable). *Hint:* It suffices to show that $(1,1)$ is in the closure of $X \setminus \{(1,1)\}$, but there is no sequence of points in $X \setminus \{(1,1)\}$ that converges to $(1,1)$.

This shows that a space with countably many elements need not be first or second countable.

25. Suppose that (X,d) is a metric space, and that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in X. Show that the sequence of real numbers $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$ (with the Euclidean metric).

26. Suppose that (X,d) is a metric space with the property that every bounded sequence converges. Prove that X is a single point.

27. (a) Prove that $\mathbb{R}$ with the Euclidean metric is a complete metric space.

(b) Prove that the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ with the Euclidean metric is **not** a complete metric space.

(c) The function

$$f : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$f(x) = \arctan(x)$$

is continuous (which you do not need to prove). Use this function to show that the continuous image of a complete metric space need not be complete.

(d) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, and that X a complete metric space. Suppose that $f : X \to Y$ is a continuous map satisfying

$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

Show that $f(X)$ is a complete metric space.

28. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathscr{C}(X,Y)$ that converges uniformly. Show that $\{f_n \mid n \in \mathbb{N}\}$ is equicontinuous.

29. For each of the following sequences of functions $f_n : \mathbb{R} \to \mathbb{R}$ in $\mathscr{C}(\mathbb{R},\mathbb{R})$, determine whether the sequence converges in the topology of pointwise convergence, in the compact-open topology, and in the topology of uniform convergence.

(a) $f_n(x) = \frac{1}{n}$

(b) $f_n(x) = \frac{x}{n}$

(c) $f_n(x) = x^n$

(d) $f_n(x) = x^2 + \frac{1}{n}$

(e) $f_n(x) = \frac{\sin(x)}{n}$

(f) $f_n(x) = \begin{cases} x, & x \in [-n, n] \\ -n, & x < -n \\ n, & x > n \end{cases}$

30. Let X and Y be topological spaces, and let $\mathscr{C}(X,Y)$ denote the space of continuous functions from X to Y in the compact-open topology.
    (a) Suppose that Y is Hausdorff. Show that $\mathscr{C}(X,Y)$ is Hausdorff.
    (b) Suppose that Y is regular. Show that $\mathscr{C}(X,Y)$ is regular.

31. For topological spaces X and Y, let $\mathscr{C}(X,Y)$ denote the space of continuous functions from X to Y in the compact-open topology. Suppose that Y is locally compact and Hausdorff. Show that the map

$$\begin{aligned} \mathscr{C}(X,Y) \times \mathscr{C}(Y,Z) &\to \mathscr{C}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

is continuous.

32. Prove the following.
    (a) A space X is compactly generated if and only if each set $B \subseteq X$ is closed in X if $B \cap C$ is closed for all compact C.
    (b) If X is a locally compact space, then X is compactly generated.
    (c) If X is a first countable space, then X is compactly generated.

33. Let X be a compactly generated space and Y a metric space. Show that $\mathscr{C}(X,Y)$ is closed in Y^X in the topology of compact convergence.

34. Let X be a space and Y a metric space.
    (a) Show that the uniform topology on Y^X is finer than the topology of compact convergence, which is finer than the topology of pointwise convergence.
    (b) Show that, if X is compact, the topology of compact convergence and the uniform topology coincide.
    (c) Show that, if X is discrete, then the topology of compact convergence and the topology of pointwise convergence coincide.

35. Show that the set of bounded functions $\mathscr{B}(\mathbb{R},\mathbb{R})$ is not closed in topology of compact convergence on $\mathbb{R}^{\mathbb{R}}$.

36. Let X be a locally compact Hausdorff space, and let Y be a space. We say that continuous maps $f, g : X \to Y$ are *homotopic* if there is a continuous map

$$h : X \times [0,1] \to Y$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$. Explain the sense in which a homotopy corresponds to a continuous map

$$H : [0, 1] \to \mathscr{C}(X, Y)$$

where $\mathscr{C}(X, Y)$ has the compact-open topology.