1. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, write "`False`". **No justification necessary.**

   (i) Let S be the set of sequence of rational numbers that are eventually zero. Then S is countable.

   True. *Hint:* Write S as a countable union of countable sets.

   (ii) Let S be the set of sequences of rational numbers that converge to zero. Then S is countable.

   False. *Hint:* Apply Cantor's diagonalization argument to the subset of S of sequences $(\frac{\epsilon_n}{n})_{n\in\mathbb{N}}$ for $\epsilon_n \in \{0,1\}$.

   (iii) Given a countable collection of uncountable sets $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$, their intersection must be uncountable.

   False. *Hint:* Consider the sets $(-\frac{1}{n}, \frac{1}{n}) \subseteq \mathbb{R}$.

   (iv) Let X be a space and $A \subseteq X$. If A is closed in X, then A will also be closed with respect to any finer topology on X.

   True. *Hint:* The complement of A will still be open in any finer topology.

   (v) Let X be a space and $A \subseteq X$. If A is closed in X, then A will also be closed with respect to any coarser topology on X.

   False. *Hint:* Consider $[0,1] \subseteq \mathbb{R}$ with respect to the standard topology and the indiscrete topology.

   (vi) Let X be a set. If $\mathcal{T}_1$ is a finer topology on X than $\mathcal{T}_2$, then the identity map on X is necessarily continuous when viewed as function from $(X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$.

   True. *Hint:* Verify that the preimage of any open set is open.

   (vii) If X is a Hausdorff space, then limits of sequences in X are unique.

   True. (Proved in class)

(viii) If X is a space where limits of sequences are unique, then X is Hausdorff.

False. *Hint:* Consider $\mathbb{R}$ with cocountable topology.

(ix) Let X be a totally ordered set with the order topology, and let $a, b \in X$. Then $[a, b]$ is the closure of (a, b).

False. *Hint:* Consider $\mathbb{N}$ in the order topology.

(x) Let $f : X \to Y$ be a homeomorphism, and let $A \subseteq X$. Then f restricts to a homeomorphism $f|_A : A \to f(A)$ between the subspaces A and $f(A)$.

True. *Hint:* Consider the definition of homeomorphism and subspace.

(xi) Two sets X and Y with the discrete topology are homeomorphic if and only if they have the same cardinality.

True. *Hint:* Any bijection is a homeomorphism.

(xii) Two sets X and Y with the cofinite topology are homeomorphic if and only if they have the same cardinality.

True. *Hint:* Any bijection is a homeomorphism.

(xiii) Let $f : X \to Y$ be a continuous injective map. If f is an open map, then f is an embedding.

True. *Hint:* Verify that f is a homeomorphism onto its image.

(xiv) Let $f : X \to Y$ be a continuous injective map. If f is a closed map, then f is an embedding.

True. *Hint:* Verify that f is a homeomorphism onto its image.

(xv) If the closure of a subspace A of a space X is connected, then $\text{Int}(A)$ is connected.

False. *Hint:* Consider $A = (0, 1) \cup (1, 2)$ as a subspace of $X = \mathbb{R}$.

(xvi) If A is a subspace of X such that $\text{Int}(A)$ is connected, then A is connected.

False. *Hint:* Consider $A = (0,1) \cup \mathbb{Q}$ as a subspace of $X = \mathbb{R}$.

(xvii) Let $f : (0,1) \to \mathbb{R}$ be the function $f(x) = \sin\left(\frac{1}{x}\right)$. Then there is no way to define f at 0 to extend f to a continuous function $f : [0,1) \to \mathbb{R}$.

True. *Hint:* If defining $f(0) = y$ resulted in a continuous function, then the union of the graph of $\sin\left(\frac{1}{x}\right)$ and the point $(0, y)$ would be path-connected.

(xviii) If X is a connected topological space, then X is also connected with respect to any coarser topology.

True. *Hint:* A separation of X in a coarser topology also constitutes a separation in a finer topology.

(xix) The space $\mathbb{R}^\omega$ in the product topology is connected.

True. *Hint:* We proved that an arbitrary product of path-connected spaces is path-connected, and hence connected.

(xx) An uncountable set with the cofinite topology is second countable.

False. (Proved in class)

(xxi) The space $\mathbb{N}$ with the cofinite topology is second countable.

True. *Hint:* There are only countably many cofinite subsets of $\mathbb{N}$.

(xxii) Let X be any set endowed with the cofinite topology. Then X is compact.

True. *Hint:* Given any open set U in any open cover, U covers all but finitely many points of X.

(xxiii) The space $\mathbb{R}^\omega$ in the product topology is normal.

True. *Hint:* It is metrizable in the product topology.

(xxiv) The space $\mathbb{R}^\omega$ in the uniform topology is normal.

True. *Hint:* It is metrizable in the uniform topology.

(xxv) Every metric space can be embedded isometrically into a complete metric space.

True. *Hint:* Homework 14 Problem 1.

(xxvi) Euclidean space $\mathbb{R}^n$ is a Baire space.

True. *Hint:* The Baire category theorem states that complete metric spaces are Baire spaces.

(xxvii) Every totally bounded space is bounded.

True. *Hint:* Show that a finite union of ϵ-balls is bounded.

(xxviii) Every compact metric space is complete.

True. *Hint:* Since a compact metric space is sequentially compact, every Cauchy sequence has a convergence subsequence, and therefore converges.

(xxix) Every complete metric space is compact.

False. *Hint:* For example, $\mathbb{R}$ is complete and non-compact.

(xxx) There exists a continuous, surjective path $[0, 1] \to [0, 1]^2$.

True. *Hint:* We outlined an existence proof for *Peano space-filling curves* in class.

(xxxi) Let X be a nonempty Baire space, and suppose that $X = \bigcup_{n\in\mathbb{N}} B_n$ for some countable collection of subsets B_n. Then $\overline{B_n}$ must have nonempty interior for some n.

True. *Hint:* $X = \bigcup_{n\in\mathbb{N}} \overline{B_n}$ and X necessarily has nonempty interior in X. Consider the definition of a Baire space.

(xxxii) The set of functions $\{f_n(x) = \frac{x}{n} \mid n \in \mathbb{N}\}$ in $\mathscr{C}(\mathbb{R}, \mathbb{R})$ is equicontinuous.

True. *Hint:*The derivatives at any point x are uniformly bounded in n by 1.

(xxxiii) The set of functions $\{f_n(x) = x^n \mid n \in \mathbb{N}\}$ in $\mathscr{C}([0,1], \mathbb{R})$ is equicontinuous.

False. *Hint:* Apply the definition of equicontinuous to the point $x_0 = 1$.

(xxxiv) A product of arbitrarily many compact spaces is compact in the product topology.

True. *Hint:* This is Tychonoff's theorem.

2. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, state a counterexample. **No justification necessary.**

Note: You can get partial credit for correctly writing "`False`" without a counterexample.

(i) There does not exist a set whose power set is countably infinite.

True. *Hint:* We proved that a set always has strictly lesser cardinality than its power set. What sets have cardinality strictly less than countable?

(ii) Let S be a subspace of a topological space X, and let $A \subseteq X$. Then the closure of $A \cap S$ is in S is equal to $\overline{A} \cap S$, where $\overline{A}$ denotes the closure of A in X.

False. For example, Let $S = \mathbb{N} \subseteq X = \mathbb{R}$, and let $A = (1, 2)$. Then $A \cap S$ and its closure in S are empty, but $\overline{A} \cap S = [1, 2] \cap S = \{1, 2\}$.

(iii) Let A be a connected subspace of a space X. Then $\overline{A}$ is connected.

True. *Hint:* See Homework #9 Problem 1.

(iv) Let X be a topological space, and $S \subseteq X$. If S is open, then $S \cap \partial S = \varnothing$.

True. *Hint:* By definition $S = \text{Int}(S)$ is disjoint from ∂S.

(v) Let X be a topological space, and $S \subseteq X$. Then $\text{Int}(S) = \text{Int}(\overline{S})$.

False. For example, consider $X = \mathbb{R}$ and $S = \mathbb{Q}$. Then $\text{Int}(\mathbb{Q}) = \varnothing$, but $\text{Int}(\overline{\mathbb{Q}}) = \text{Int}(\mathbb{R}) = \mathbb{R}$.

(vi) Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then $\overline{X \setminus A} = X \setminus \overline{A}$.

False. For example, $A = [0, \infty)$ as a subset of $X = \mathbb{R}$ with the Euclidean metric. Then $0 \in \overline{X \setminus A} = (-\infty, 0]$ but $0 \notin X \setminus \overline{A} = (-\infty, 0)$.

(vii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X has the discrete topology, then so does the subspace $f(X) \subseteq Y$.

False. Let $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the indiscrete topology. Let $f : X \to Y$ be the identity map on $\mathbb{R}$. Then f is continuous and X has the discrete topology, but $f(X) = \mathbb{R}$ does not.

(viii) Every Hausdorff space is metrizable.

False. For example, we proved that the box topology on $\mathbb{R}^\omega$ is Hausdorff but not metrizable.

(ix) Let $\{A_i\}_{i \in I}$ be a collection of path-connected subspaces of a space X, such that $\bigcap_{i \in I} A_i \neq \varnothing$. Then $\bigcup_{i \in I} A_i$ is path-connected.

True. *Hint:* Fix a point $x \in \bigcap_{i \in I} A_i$; it must be in the same path-component of any point $y \in A_i$.

(x) If X is a metric space, then every compact subset of X is closed and bounded.

True. *Hint:* Verify that our proof for $\mathbb{R}^n$ holds in a general metric space.

(xi) If X is a metric space, then every closed and bounded subset of X is compact.

False. For example, $[-\pi, \pi] \cap \mathbb{Q}$ is a closed, bounded, and non-compact subset of $\mathbb{Q}$.

(xii) If X is a limit point compact space, then X is also limit point compact with respect to any coarser topology.

True. *Hint:* Consider the definition of a limit point.

(xiii) Let $f : X \to Y$ be a continuous function from a limit point compact space X to a space Y. Then $f(X)$ is limit point compact.

False. Let $X = \mathbb{N} \times \{0, 1\}$, the product of the discrete space $\mathbb{N}$ and the indiscrete space $\{0, 1\}$. We proved in class that X is limit point compact. Let $f : X \to \mathbb{N}$ be the projection onto the first coordinate. Then f is surjective, but its image $\mathbb{N}$ is a non-compact metric space, and therefore not limit point compact.

(xiv) A closed subset $A \subseteq X$ of a limit point compact space X is also limit point compact.

True. *Hint:* Let B be an infinite subset of A. By assumption it has a limit point x in X. Show that since $B \subseteq A$ this point x must also be a limit point of A, and since A is closed, $x \in A$.

(xv) Any second countable space X is separable.

True. See Homework 12 Problem 4(a).

(xvi) Every metrizable space is second countable.

False. For example, the discrete topology on $\mathbb{R}$ is metrizable but not second countable.

(xvii) Suppose that $(X, \mathcal{T})$ is a topological space that is first countable. Then any coarser topology on X will also be first countable.

False. For example, the discrete topology on $\mathbb{R}$ is first countable. The cofinite topology on $\mathbb{R}$ is coarser, but is not first countable.

(xviii) Suppose that $(X, \mathcal{T})$ is a topological space that is first countable. Then any finer topology on X will also be first countable.

False. For example, the indiscrete topology on $\mathbb{R}$ is first countable. The cofinite topology on $\mathbb{R}$ is finer, but is not first countable.

(xix) A subspace of a second countable space is second countable.

True. *Hint:* If $Y \subseteq X$ and $\mathcal{B}$ is a countable basis for X, consider $\{B \cap Y \mid B \in \mathcal{B}\}$.

(xx) A product of Lindelöf spaces is Lindelöf.

False. You proved that the Sogenfrey line $\mathbb{R}_\ell$ is Lindelöf, but we saw in class that its product $\mathbb{R}_\ell^2$ is not.

(xxi) The continuous image of a normal space is normal.

False. For example, the standard topology on $\mathbb{R}$ is normal (since it is metrizable). The identity map $(\mathbb{R}, \text{standard}) \to (\mathbb{R}, \text{cofinite})$ is continuous since the standard topology is finer. But the cofinite topology on $\mathbb{R}$ is not normal; it is not even Hausdorff.

(xxii) A space X is locally compact and Hausdorff if and only if it is homeomorphic to an open subset of a compact Hausdorff space.

True. *Hint:* If X is locally compact and Hausdorff, then its one-point compactification is defined and is Hausdorff.

(xxiii) Let $f : X \to Y$ be a continuous map of metric spaces, and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then $\Big(f(x_n)\Big)_{n\in\mathbb{N}}$ is a Cauchy sequence.

False. For example, let $X = Y = (0, \infty)$, and let $f(x) = \frac{1}{x}$. Then the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is Cauchy, but its image $(n)_{n\in\mathbb{N}}$ is not.

(xxiv) A subspace S of $\mathbb{R}^n$ with the Euclidean metric is complete if and only if it is closed.

True. *Hint:* If S is closed, then it must contain the limit of any Cauchy sequence converging to a point of X. Conversely, if S is not closed, choose a point of $\overline{S} \setminus S$ and consider any sequence converging to this point.

(xxv) Every totally bounded metric space is compact.

False. For example, $(0,1)$ is totally bounded but non-compact.

3. Find the interior, closure, and boundary of each of the following subsets of $\mathbb{R}^\omega$, with respect to
   (i) the box topology, (ii) the uniform topology, (iii) the product topology.

   (a) the set of all bounded sequences

   (b) the set of all constant sequences

   Solution: Let C be the set of all constant sequences.
   In the box topology, C is a closed set with empty interior, so $C = \overline{C} = \partial C$, and $\text{Int}(C) = \varnothing$. To see this, first note that every basis element $\prod_n U_n$ for the box topolgy, with $U_n \subseteq \mathbb{R}$ open, necessarily contains non-constant sequences; simply construct a sequence by choosing any point x_1 in U_1, and any point x_2 in $U_2 \setminus \{x_1\}$, and all other points x_n arbitrarily. This observation shows that C has empty interior.
   Next, we observe that the complement of C is open. Let $(x_n)_{n\in\mathbb{N}}$ be any non-constant sequence, so there is some N and M so that $x_N \neq x_M$. Because $\mathbb{R}$ is Hausdorff we can choose disjoint neighbourhoods U_N of x_N and U_M of x_M. Then every sequence in the following neighbourhood is nonconstant, since its N^{th} and M^{th} terms must differ:

$$\prod_n U_n \qquad \text{with} \qquad U_n = \begin{cases} \mathbb{R}, & n \neq N, M \\ U_N, & n = N \\ U_M, & n = M. \end{cases}$$

   This proves that $(x_n)_{n\in\mathbb{N}}$ is an interior point of the complement of C, and so C is closed. Thus $C = \overline{C}$, and $\partial C = \overline{C} \setminus \text{Int}(C) = C \setminus \varnothing = C$.
   The above argument also carries over to the product topology, by considering basis elements for the product topology in place of basis elements for the box topology. In the product topology, too, $C = \overline{C} = \partial C$, and $\text{Int}(C) = \varnothing$.
   Finally, we can verify that $C = \overline{C} = \partial C$, and $\text{Int}(C) = \varnothing$ in the uniform topology. Since the complement of C is open in the product topology, and the uniform topology is finer, it must be open in the uniform topology. Thus C is closed. If C contained an open subset in the uniform topology, then the same subset would be open in the box topology, which is finer – this is impossible since $\text{Int}(C)$ is empty in the box topology. Hence $\text{Int}(C) = \varnothing$ in the unfirom topology, and $C = \overline{C} = \partial C$.

   (c) the set of all non-constant sequences

   Solution: Let N be the set of all non-constant sequences. We proved above that, in any topology, N is open, and moreover that its boundary – which is equal to the boundary of its complement – is the set of all constant sequences C. Hence $\text{Int}(N) = N$, $\partial N = C$, and $\overline{N} = \mathbb{R}^\omega$.

(d) the set of all sequences that are eventually constant
(e) the set of all sequences of rational numbers
(f) the set of all sequences of natural numbers
(g) the set of all sequences of strictly positive numbers
(h) the set of all sequences of nonnegative numbers

4. Let (X, d) be a metric space, and let $A \subseteq X$. For $x \in X$, define

$$d(x, A) = \inf_{a \in A} d(x, a).$$

(a) Prove that following the "distance to A" function is a continuous function on X,

$$d_A : X \to \mathbb{R}$$
$$d_A(x) = d(x, A).$$

Solution: Let $V \subseteq \mathbb{R}$ be any open set. If $d_A^{-1}(V)$ is empty, then it is open. So suppose that $x \in d_A^{-1}(V)$; we will show that x is an interior point of $d_A^{-1}(V)$, and so conclude that $d_A^{-1}(V)$ is open. Since $d_A(x) \in V$ and V is open, there must exist some ϵ so that $B_\epsilon(d_A(x)) \subseteq V$. Let $U = B_{\frac{\epsilon}{2}}(x)$. Then for any $u \in U$ and $a \in A$,

$$d(u, a) \leq d(u, x) + d(x, a) < \frac{\epsilon}{2} + d(x, a)$$

Taking the infimum over all elements $a \in A$, we conclude

$$d_A(u) = \inf_{a \in A} d(u, a) \leq \inf_{a \in A} \left(\frac{\epsilon}{2} + d(x, a) \right) = \frac{\epsilon}{2} + d_A(x).$$

Furthermore, for any $u \in U$ and $a \in A$,

$$d(a, x) \leq d(a, u) + d(u, x) < d(a, u) + \frac{\epsilon}{2}$$

Again taking the infimum of both sides of the inequality over all $a \in A$, we find

$$d_A(x) \leq d_A(u) + \frac{\epsilon}{2}$$

so

$$d_A(u) \geq d_A(x) - \frac{\epsilon}{2}.$$

Combining these inequalities, we conclude that $d_A(u) \in B_{\frac{\epsilon}{2}}(d_A(x))$ for any $u \in U$. This implies that $U \subseteq d_A^{-1}(B_\epsilon(d_A(x))) \subseteq d_A^{-1}(V)$. We conclude that $d_A^{-1}(V)$ is open, and so d_A is continuous, as claimed.

(b) Suppose that A is compact. Let $x \in X$. Show that there is some $a \in A$ so that $d(x, A) = d(x, a)$.

Solution: Fix $x \in X$, and define a function

$$d_x : A \to \mathbb{R}$$
$$d_x(a) = d(x, a).$$

Since A is compact, by the Extreme Value Theorem (Homework 11 Problem 3(c)), the function d_x achieves its infimum at some point $a \in A$. But its infimum is precisely the definition of the value $d(x, A)$. Hence $d(x, A) = d(x, a)$.

(c) Let A be any subset of X. Show that $\overline{A} = \{x \mid d(x, A) = 0\}$.

Solution: Suppose first that x satisfies $d(x, A) = r > 0$. This means that $d(x, a) \geq r$ for all $a \in A$. Hence the ball $B_r(x)$ is disjoint from A, and we conclude that x is not in $\overline{A}$.
Suppose conversely that $d(x, A) = \inf_{a \in A} d(x, a) = 0$. This means that, for all $\epsilon > 0$, there exists some $a \in A$ so that $d(x, a) < \epsilon$. In other words, for all $\epsilon > 0$, there is some element $a \in A$ contained in the ball $B_\epsilon(x)$. Hence $x \in \overline{A}$.

5. (a) Let X and Y be spaces, and let X^* and Y^* be partitions of X and Y, respectively, with the quotient topology. Under what conditions will a continuous map $f : X \to Y$ induce a well-defined, continuous map $f^* : X^* \to Y^*$?

   (b) Let T denote the quotient of $\mathbb{R}$ by the equivalence relation $x \sim 2\pi x$; T is homeomorphic to the unit circle. Which continuous functions $f : \mathbb{R} \to \mathbb{R}$ descend to continuous functions $f^* : T \to T$?

6. Let X_i, $i \in I$ be a collection of topological spaces, and let $X = \prod_{i \in I} X_i$ denote their product in the box topology. Show that a sequence $((x_i^n)_{i \in I})_{n \in \mathbb{N}}$ of points in X converges if and only if it satisfies the following: Firstly, it converges pointwise at each index $i \in I$. In other words, for each $i \in I$ the sequence $(x_i^n)_{n \in \mathbb{N}}$ in X_i converges. Secondly, for all but finitely many i, the sequence $(x_i^n)_{n \in \mathbb{N}}$ must be eventually constant.

7. Suppose that there are embeddings $X \to Y$ and $Y \to X$. Show by example that this does not imply that X and Y are homeomorphic.

Solution: Let $X = (0, 1)$ and $Y = [0, 1]$. We have proved that X is noncompact and Y is compact, and so in particular that X and Y are not homeomorphic.

However, X embeds in Y as the subspace $(0, 1) \subseteq [0, 1]$. We have also seen that Y is homeomorphic to the subspace $[\frac{1}{4}, \frac{3}{4}] \subseteq X$. This homeomorphism defines an embedding $Y \hookrightarrow X$. Thus there exist embeddings of Y into X and of X into Y, even though X and Y are not homeomorphic.

8. **Definition (Convex subsets of $\mathbb{R}^n$).** Let A be a subset of $\mathbb{R}^n$ (with the Euclidean metric). Then A is called *convex* if $t\mathbf{x} + (1-t)\mathbf{y} \in A$ for every $\mathbf{x}, \mathbf{y} \in A$ and any $t \in [0,1]$.

Prove that any convex subset of $\mathbb{R}^n$ is connected.

Solution: Let $A \subseteq \mathbb{R}^n$ be a convex set. Then, given any $\mathbf{x} = (x_i), \mathbf{y} = (y_i) \in A$, consider the function

$$\gamma : [0,1] \to A$$
$$\gamma(t) = t\mathbf{x} + (1-t)\mathbf{y}$$

Then γ is continuous, since its i^{th} coordinate function $\gamma(t)_i = tx_i + (1-t)y_i$ is linear and hence continuous. Moreover, $\gamma(0) = \mathbf{y}$ and $\gamma(1) = \mathbf{x}$, so γ is a path from $\mathbf{y}$ to $\mathbf{x}$. We conclude that A is path-connected, and therefore connected.

9. (a) Show that the unit interval $[0,1]$ cannot be partitioned into a countable union of more than one closed sets. *Hint:*
   - (i) Show that $[0,1]$ cannot be partitioned into a finite number (of at least two) closed sets $C_1, C_2, \ldots, C_n$.
   - (ii) Suppose that $[0,1]$ were partitioned into a countably infinite number of closed sets $\{C_n\}_{n\in\mathbb{N}}$. Show that each C_n must have nonempty boundary.
   - (iii) Let $B = \bigcup_{n\in\mathbb{N}} \partial C_n$. Show that, for each n, the boundary ∂C_n has empty interior as a subspace of B.
   - (iv) Apply the Baire category theorem to reach a contradiction.

Solution. First suppose that $[0,1]$ were partitioned into a finite number (of at least two) closed sets $C_1, C_2, \ldots, C_n$. But then $C = C_2 \cup \cdots \cup C_n$ is nonempty and closed, so C_1 and C constitute a separation of $[0,1]$. This is a contradiction.

Next suppose that $[0,1]$ were partitioned into a countably infinite number of closed sets $\{C_n\}_{n\in\mathbb{N}}$. Recall that this means that the sets C_n are nonempty, pairwise disjoint, and their union is $[0,1]$. Note that the set C_n cannot also be open, or else C_n and its complement would constitute a separation of $[0,1]$. This implies that for each n, C_n has nonempty boundary.

So consider the set $B = \bigcup_{n\in\mathbb{N}} \partial C_n$. Because this is a partition, $B = [0,1] \setminus \bigcup_{n\in\mathbb{N}} \mathrm{Int}(C_n)$, and hence B is closed. Since $B \subseteq [0,1]$, it is a compact, Hausdorff space, and hence a Baire space by Homework # 12.

We claim that, for each n, ∂C_n has empty interior as a subspace of B. Given a neighbourhood U of any point $b \in \partial C_n$ in $[0,1]$, by definition of boundary, the set U must intersect some disjoint set C_m, and (again by definition of the boundary) it must intersect C_m in a point in ∂C_m. Hence the neighbourhood $U \cap B$ of b in B intsersects ∂C_m, so b is not an interior point of ∂C_n, and ∂C_n has empty interior.

This is a contradiction: On one hand, B is a union of countably many subsets ∂C_n with emtpy interior. On the other hand, since B is a Baire space, this union must have empty interior in B, so cannot be equal to B. Hence, we conclude that we cannot partition the interval $[0, 1]$ into a countably infinite union of closed subsets.

(b) Consider the natural numbers $\mathbb{N}$ with the cofinite topology. Show that $\mathbb{N}$ is locally connected but not locally path-connected.

Solution. We first make the following observation. Let $n \in \mathbb{N}$, and let U be any neighbourhood of n. Then U is a countable subset of $\mathbb{N}$ with finite complement. We claim that the subspace topology on U is also the cofinite topology on U. To see this, consider an open subset of U; by definition this open subset must have the form $U \cap V$ for some open subset V of $\mathbb{N}$. But since both U and V have finite complements, their intersection $U \cap V$ has a finite complement in $\mathbb{N}$. Hence $U \cap V$ has a finite complement in U. Conversely, given any cofinite subset W of U, then W is also cofinite in $\mathbb{N}$, hence open in both $\mathbb{N}$ and in U.

We first show that $\mathbb{N}$ is locally connected. For $n \in \mathbb{N}$, let U be any neighbourhood of $\mathbb{N}$. This means that U has a finite complement in $\mathbb{N}$. It suffices to show that U is connected. Suppose otherwise; suppose that $A \cup B$ were a separation of U for open subsets $A, B \subseteq U$. By the above observation, the complement of A in U is finite. But by assumption this complement is B, which contradicts the assumption that B is open in U. Hence no separation exists, and we conclude that U is connected.

Next we show that $\mathbb{N}$ is not locally path connected. Let $n \in \mathbb{N}$, and let U be any neighbourhood of $\mathbb{N}$. Suppose for contradiction that $\gamma : [0, 1] \to U$ were any nonconstant path. Then γ induces a partition of U into preimages

$$\left\{ \gamma^{-1}(\{x\}) \;\middle|\; x \in \gamma([0, 1]) \right\}.$$

Since γ is continuous and $\{x\}$ is closed in U, this is a partition of $[0, 1]$ into closed subsets. The image $\gamma([0, 1])$ is necessarily countable (being a subset of $\mathbb{N}$), so this is a partition of $[0, 1]$ into a countable union of disjoint closed sets. By part (a), this partition must consist of a single closed set, which implies that γ is the constant function. Since U has more than one point, and any path into U is constant, we conclude that U is not path-connected.

10. (a) Prove that any path-connected space is connected.

(b) Prove that any locally path-connected space is locally connected.

(c) State an example of a space that is not connected or path-connected

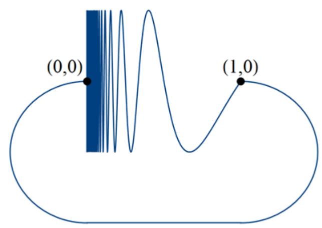
Solution. The discrete topology on two points is not connected or path-connected.

(d) State an example of a space that is connected but not path-connected.

Solution. The topologists sine curve is connected but not path-connected.

(e) State an example of a space that is connected and path-connected but not locally connected or locally path-connected

Solution. Consider the following modification of the topologist sine curve: we can add an arc connecting the point $(0,0)$ to the point $(1,0)$, as shown. The resulting space now has a single path component, and is therefore path connected – but it still fails to be locally connected (and hence locally path-connected) at $(0,0)$.



(f) State an example of a space that is connected and locally connected, but not path-connected or locally path-connected.

Solution. The arguments made in Problem 9 show that ($\mathbb{N}$, cofinite) is connected and locally connected, but not path-connected or locally path-connected.

(g) State an example of a space that is connected, path-connected, locally connected, but not locally path-connected.

Hint: We can modify the space ($\mathbb{N}$, cofinite) to create a space that is path-connected, and is still locally connected, but still fails to be locally path-connected. Take the product of $[0,1]$ (with the standard topology) and ($\mathbb{N}$, cofinite), and then quotient by the equivalence relation that identifies all the points of the form $(1,n)$ to a single point. This construction is called the *cone* on ($\mathbb{N}$, cofinite). It is an exercise to verify that this space is still locally connected but not locally path connected.

(h) State an example of a space that is locally connected, but not locally path-connected, connected, or path-connecterd.

Solution. Let $\{0,1\}$ be the discrete space on two points. Then the product of $\{0,1\}$ and ($\mathbb{N}$, cofinite) is locally connected, but not locally path-connected, connected, or path-connected.

11. Let $\mathbb{R}_K$ denote $\mathbb{R}$ with the K-topology. In this problem we will show that $\mathbb{R}_K$ is connected but not path-connected.

(a) Explain why $(-\infty,0)$ and $(0,\infty)$ inherit their usual topologies as subsaces of $\mathbb{R}_K$.

(b) Show that $\mathbb{R}_K$ is connected.

Solution. Suppose that $A \cup B$ were a separation of $\mathbb{R}_K$ by open subsets A and B. By part (a), the subspace topology $(-\infty, 0)$ and $(0, \infty)$ is the standard topology, so both subsets are connected. This means that $(-\infty, 0)$ must be contained entirely in either A or B (let's say A), and that $(0, \infty)$ must be contained entirely in either A or B.
This means that the only possibilities for A, B are $(0, \infty), [0, \infty)$ or $[0, \infty), (0, \infty)$, or $(0, \infty) \cup (0, \infty), \{0\}$. But, in all cases, the set containing 0 is not open in $\mathbb{R}_K$, and so this is not a valid separation.

(c) Show that any interval containing $[0, 1] \subseteq \mathbb{R}_K$ is non-compact.

Hint: Let I be an interval containing $[0, 1]$. Then consider the open cover

$$\{I \setminus K\} \cup \left\{ \left(\frac{1}{n}, 1 \right) \;\middle|\; n \in \mathbb{N}, n \geq 2 \right\}.$$

(d) Show that $\mathbb{R}_K$ is not path-connected.

Solution. Suppose that $\gamma : [0, 1] \to \mathbb{R}_K$ were a path from 0 to 1. Since $[0, 1]$ is connected, the image $\gamma([0, 1])$ must be connected. It follows that $\gamma([0, 1])$ must also be connected in the standard topology: since the K-topology is finer than the standard topology, any separation in the standard topology would also constitute a sepration in the K-topology. The image $\gamma([0, 1])$ is therefore an interval containing $[0, 1]$.
But $[0, 1]$ is compact, so the image $\gamma([0, 1])$ must also be compact. This contradicts part (c). We conclude that no path exists, and the K-topology is not path-connected.

12. (a) Let A and B be compact subspaces of a topological space X. Show that $A \cap B$ need not be compact. *Hint:* Consider $\mathbb{R}$ with the topology $\{U \mid 0, 1 \notin U\} \cup \{\mathbb{R}\}$.

False. Consider $\mathbb{R}$ with the topology $\{U \mid 0, 1 \notin U\} \cup \{\mathbb{R}\}$. Then the subspaces $\{0\} \cup \{n \mid n \in \mathbb{N}, n \geq 2\}$ and $\{1\} \cup \{n \mid n \in \mathbb{N}, n \geq 2\}$ are both compact, since any open cover of either must contain the open set $\mathbb{R}$. Their intersection $\{n \mid n \in \mathbb{N}, n \geq 2\}$, however, is not compact, since it is an infinite discrete subspace.

(b) Let A and B be compact subspaces of a Hausdorff space X. Show that $A \cap B$ is compact.

13. In this problem, we will show that **any** nonempty set admits a topology with respect to which it is compact and Hausdorff. Specifically, let X be a nonempty set, and let x_0 be a distinguished element of X. Let

$$\mathcal{T} = \{A \subseteq X \mid x_0 \notin A \text{ or } X \setminus A \text{ is finite }\}.$$

(a) Show that $\mathcal{T}$ defines a topology on X.

Solution. We verify the three axioms.
First note that $X \setminus X = \varnothing$ is finite, so $X \in \mathcal{T}$, and $x_0 \notin \varnothing$, so $\varnothing \in \mathcal{T}$.
Next, consider sets A and B in $\mathcal{T}$. If one or both of A and B do not contain x_0, then their intersection $A \cap B$ does not contain x_0, so $A \cap B \in \mathcal{T}$. Otherwise, both $X \setminus A$ and $X \setminus B$ must be finite. But then by de Morgan's law,

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B),$$

so $X \setminus (A \cap B)$ is finite and $A \cap B \in \mathcal{T}$.
Finally, let $\{A_i\}_{i \in I}$ be an arbitrary collection of elements of $\mathcal{T}$. If no element of A_i contains x_0, then their union $\bigcup_{i \in I} A_i$ will not contain x_0, so $\bigcup_{i \in I} A_i \in \mathcal{T}$. If A_i contains x_0 of some i, then $X \setminus A_i$ is finite. Then the complement of the union $\bigcup_{i \in I} A_i$ will be contained in the complement of A_i, and hence also be finite. We conclude that the union $\bigcup_{i \in I} A_i$ is contained in $\mathcal{T}$.

(b) Verify that $(X, \mathcal{T})$ is Hausdorff.

Solution. Let $x, y \in X$ be two distinct points. If neither x nor y is equal to x_0, then $\{x\}$ and $\{y\}$ are disjoint open neighbourhoods of x and y, respectively. If $x = x_0$, then $X \setminus \{y\}$ and $\{y\}$ are disjoint open neighbourhoods of x_0 and y, respectively. Thus X is Hausdorff.

(c) Verify that $(X, \mathcal{T})$ is compact.

Solution. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover of X. Since $\mathcal{U}$ is a cover, some element U_{j_0} must contain x_0. But then U_{j_0} must be cofinite. Suppose $X \setminus U_{j_0} = \{x_1, x_2, \ldots, x_n\}$. Let $U_{j_i} \in \mathcal{U}$ be an open set containing x_i. Then

$$U_{j_0}, U_{j_1}, \ldots, U_{j_n}$$

is a finite subcover of $\mathcal{U}$. We conclude that X is compact.

14. Let $p : X \to Y$ be a closed, continuous, surjective map, with the property that $p^{-1}(y)$ is compact for all $y \in Y$. Suppose that Y is compact. Prove that X is compact.

15. Let X be compact and Y be Hausdorff. Then any continuous bijection $f : X \to Y$ is a homeomorphism.

Solution. We must show that f^{-1} is continuous. It suffices to show that f is a closed map, which will imply that the inverse image of a closed subset under f^{-1} is closed. So consider a closed subset $C \subseteq X$. Since X is compact, C must be compact. Since Y is continuous, $f(C)$ is compact in Y. But Y is Hausdorff, so $f(C)$ must be closed.

16. Let $X \neq \varnothing$ be a compact Hausdorff space. Show that if X has no isolated points, then X is uncountable.

    Hint: See Munkres Theorem 27.7.

17. Show that a connected metric space with at least 2 points must have uncountably many points.

    Hint: Let d denote the metric on X. Suppose that $x, y \in X$. Show that, for all $r \in [0, d(x,y)]$, there must be some element $z \in X$ with $d(x,z) = r$, or X would be disconnected.

18. Let $X = [0,1]^\omega$ in the uniform topology. Show that X is not limit point compact, and therefore not compact.

    Hint: Consider the subset of X of sequences of the form $(\epsilon_n)_{n\in\mathbb{N}}$ where $\epsilon_n \in \{0,1\}$ for all n.

19. Determine where $\mathbb{R}^\omega$ with the box topology is first countable, or second countable.

    Solution. The box topology is not first countable (and therefore not second countable). For a point $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\omega$, suppose that

$$U_1 = U_{1,1} \times U_{1,2} \times U_{1,3} \times \cdots$$
$$U_2 = U_{2,1} \times U_{2,2} \times U_{2,3} \times \cdots$$
$$U_3 = U_{3,1} \times U_{3,2} \times U_{3,3} \times \cdots$$

    were a countable local neighbourhood basis at x. Since $U_{i,j}$ is open, there is some ball $B_{r_{i,j}}(x_j) \subseteq U_{i,j}$. Now consider the neighbourhood V of x defined by

$$V = B_{\frac{1}{2}r_{1,1}}(x_1) \times B_{\frac{1}{2}r_{2,2}}(x_2) \times B_{\frac{1}{2}r_{3,3}}(x_3) \times \cdots$$

    The open set V does not contain U_n for any n, since $U_{n,n} \not\subseteq B_{\frac{r_{n,n}}{2}}(x_n)$ by construction. This contradicts the assumption that $\{U_n\}$ is a local basis at x.

20. Let $\mathbb{R}_\ell$ denote $\mathbb{R}$ with the *lower limit topology*, the topology generated by the basis $\mathcal{B} = \{[a,b) \mid a, b \in \mathbb{R}\}$. This space is called the *Sorgenfrey line.*
    (a) Show that the lower limit topology is finer than the standard topology on $\mathbb{R}$.
    (b) Show that $\mathbb{R}_\ell$ is Hausdorff.
    (c) Show that $\mathbb{R}_\ell$ is first countable and separable.

(d) Show that $\mathbb{R}_\ell$ is not second countable.

(e) Show that $\mathbb{R}_\ell$ is not metrizable.

(f) Show that $[0,1] \subseteq \mathbb{R}_\ell$ is not limit point compact.

21. Let X be a separable metrizable space. Prove that any subspace $A \subseteq X$ is also separable.

Solution. We proved that, since X is metrizable, being separable is equivalent to being second countable. So X has a countable basis $\mathcal{B}$. But, we proved moreover that if $\mathcal{B}$ is a basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace A. Hence A is also second countable, and we conclude that A is separable.

22. Let X be a regular space. Show that every pair of points in X have neighbourhoods with disjoint closures.

23. Let $X = \prod_{i \in I} X_i$ be a product of nonempty spaces.

(a) Suppose X is regular. Show that X_i is regular for all i.

(b) Suppose X is normal. Show that X_i is normal for all i.

24. **Definition (Arens–Fort space).** Let $X = \mathbb{N} \times \mathbb{N}$ with the following topology,

$$\{\, U \mid (1,1) \notin U\} \cup \left\{ U \;\middle|\; \begin{array}{l} (1,1) \in U, \text{ and } U \text{ contains all but a finite} \\ \text{number of points in all but a finite number of colums} \end{array} \right\}.$$

This space is called *Arens–Fort space*.

Recall that a *column* of $\mathbb{N} \times \mathbb{N}$ is a subset of form $\{(m,n) \mid n \in \mathbb{N}\}$ for some fixed m.

(a) Show that X is Hausdorff.

(b) Show that X is regular.

(c) Show that X is not compact.

(d) Show that X is not first countable (and therefore not second countable or metrizable). *Hint:* It suffices to show that $(1,1)$ is in the closure of $X \setminus \{(1,1)\}$, but there is no sequence of points in $X \setminus \{(1,1)\}$ that converges to $(1,1)$.

This shows that a space with countably many elements need not be first or second countable.

25. Suppose that (X,d) is a metric space, and that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in X. Show that the sequence of real numbers $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$ (with the Euclidean metric).

26. Suppose that (X,d) is a metric space with the property that every bounded sequence converges. Prove that X is a single point.

Solution. We will prove the contrapositive: in a metric space with at least two points, there are sequences that are bounded but do not converge. Suppose that (X,d) is a metric space and that x and y are two distinct points in X. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ defined by

$$x\ y\ x\ y\ x\ y\ x\ y \cdots$$

This sequence is bounded, since (for example) every term is contained in the ball $B_{2d(x,y)}(x)$. But we will show that the sequence does not converge. Let $z \in X$ be any candidate limit point. Since X is Hausdorff, if z is not equal to x, we can find a neighbourhood U of z that does not contain x. If $z = x$, we can find a neighbourhood V of x that does not contain y. Then the sequence cannot converge to z, since in both cases we have a neighbourhood of z that does not contain a_n for infinitely many values of n. Hence we have a bounded and non-convergent sequence.

27. (a) Prove that $\mathbb{R}$ with the Euclidean metric is a complete metric space.

 (b) Prove that the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ with the Euclidean metric is **not** a complete metric space.

 (c) The function

$$f : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$f(x) = \arctan(x)$$

 is continuous (which you do not need to prove). Use this function to show that the continuous image of a complete metric space need not be complete.

 (d) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, and that X a complete metric space. Suppose that $f : X \to Y$ is a continuous map satisfying

$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

 Show that $f(X)$ is a complete metric space.

28. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathscr{C}(X, Y)$ that converges uniformly. Show that $\{f_n \mid n \in \mathbb{N}\}$ is equicontinuous.

29. For each of the following sequences of functions $f_n : \mathbb{R} \to \mathbb{R}$ in $\mathscr{C}(\mathbb{R}, \mathbb{R})$, determine whether the sequence converges in the topology of pointwise convergence, in the compact-open topology, and in the topology of uniform convergence.

 (a) $f_n(x) = \frac{1}{n}$

 (b) $f_n(x) = \frac{x}{n}$

Solution. The sequence $f_n(x) = \frac{x}{n}$ converges pointwise and uniformly on compact sets to the limit $f(x) = 0$, but it does not converge uniformly. To check pointwise convergence, fix a point $x_0 \in \mathbb{R}$. Then

$$\lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} \frac{x_0}{n} = 0,$$

so the sequence converges to 0 at every piont $x_0 \in X$.

To verify uniform convergence on compact sets, let $C \subseteq \mathbb{R}$ be a compact subset. Then C is bounded, so C is contained in an interval $[-N, N]$ for some $N \in \mathbb{R}$. Fix $\epsilon > 0$. Then choose n large enough so that $\frac{N}{n} < \epsilon$. Then for all $x \in C \subseteq [-N, N]$,

$$|f_n(x)| = \frac{|x|}{n} \leq \frac{N}{n} < \epsilon,$$

and so the sequence $f_n|_C$ converges uniformly to zero.

Finally, we will show that uniform convergence fails. Since uniform convergence implies pointwise convergence, the only possible uniform limit is the pointwise limit $f(x) = 0$. Take $\epsilon = 1$. Then for any $n \in \mathbb{N}$, we can choose $x > n$, so that $f_n(x) > 1$. Hence f_n is not contained in the 1-ball of f in the uniform metric, and the sequence fails to converge uniformly.

(c) $f_n(x) = x^n$

(d) $f_n(x) = x^2 + \frac{1}{n}$

(e) $f_n(x) = \frac{\sin(x)}{n}$

(f) $f_n(x) = \begin{cases} x, & x \in [-n, n] \\ -n, & x < -n \\ n, & x > n \end{cases}$

30. Let X and Y be topological spaces, and let $\mathscr{C}(X,Y)$ denote the space of continuous functions from X to Y in the compact-open topology.

(a) Suppose that Y is Hausdorff. Show that $\mathscr{C}(X,Y)$ is Hausdorff.

Solution. Let f and g denote two distinct elements of $\mathscr{C}(X,Y)$. Since $f \neq g$, there is some point $x_0 \in X$ so that $f(x_0) \neq g(x_0)$. But then since Y is Hausdorff, the points $f(x)$ and $g(x)$ have disjoint neighbourhoods U_f and U_g, respectively. The set $\{x_0\}$ is finite and therefore compact. So consider the two subbasis elements for the compact-open topology on $\mathscr{C}(X,Y)$ given by

$$S(x_0, U_f) = \{h \mid h(x_0) \in U_f\} \qquad S(x_0, U_g) = \{h \mid h(x_0) \in U_g\}$$

Then $f \in S(x_0, U_f)$ and $g \in S(x_0, U_g)$ by construction. Moreover, these open sets are disjoint, since for any function h, the value $h(x_0)$ can be contained in at most one of the two disjoint sets $S(x_0, U_f)$ and $S(x_0, U_g)$.

(b) Suppose that Y is regular. Show that $\mathscr{C}(X,Y)$ is regular.

Solution. We must first check that $\mathscr{C}(X,Y)$ is a T_1-space. But note that Y is regular, and therefore Hausdorff, so $\mathscr{C}(X,Y)$ is Hausdorff by part (a), and hence T_1.

Now it suffices to check that, for any function $f \in \mathscr{C}(X,Y)$ and neighbourhood $\mathscr{U}$ of f, there is some neighbourhood $\mathscr{V}$ of f with $\overline{\mathscr{V}} \subseteq \mathscr{U}$. We claim that it is enough to check this property for subbasis elements $\mathscr{U} = S$.

Suppose this property holds for all elements of a subbasis of $\mathscr{C}(X,Y)$, and let $\mathscr{U}$ be an arbitrary open set containing f. Since finite intersections of subbasis elements form a basis for $\mathscr{C}(X,Y)$, there is intersection $S_1 \cap \ldots \cap S_n$ of subbasis elements with $f \in S_1 \cap \ldots \cap S_n \subseteq \mathscr{U}$. Then for each i, there is some open set $\mathscr{V}_i$ with $f \in \mathscr{V}_i$ and $\overline{\mathscr{V}_i} \in\subseteq S_i$. Consider the intersection $\mathscr{V}_1 \cap \ldots \cap \mathscr{V}_n$. It must contain f by construction. Moverover, since $\overline{\mathscr{V}_i}$ is a closed set containing $\mathscr{V}_1 \cap \ldots \cap \mathscr{V}_n$, the closure $\overline{\mathscr{V}_1 \cap \ldots \cap \mathscr{V}_n}$ must be contained

in $\overline{\mathscr{V}_1} \cap \cdots \cap \overline{\mathscr{V}_n} \subseteq S_1 \cap \ldots \cap S_n \subseteq U$. Thus $\mathscr{V}_1 \cap \ldots \cap \mathscr{V}_n$ is the desired sub-neighbourhood of f.
So we will prove the result for the subbasis

$$\{S(C,U) \mid \; C \subseteq X \text{ compact}, U \subseteq Y \text{ open}\}$$

for $\mathscr{C}(X,Y)$. Let $f \in \mathscr{C}(X,Y)$, and let $S(C,U)$ be a neighbourhood of f. Suppose that V is an open set such that $\overline{V} \subseteq U$. We claim that $\overline{S(C,V)} \subseteq S(C,U)$.
Let $g \in \overline{S(C,V)}$. We wish to show that $g \in S(C,U)$, that is, $g(C) \subseteq U$. Suppose otherwise; then there is some $x \in C$ with $g(x) \notin (Y \setminus U) \subseteq (Y \setminus \overline{V})$. This shows that $g \in S(\{x\}, Y \setminus \overline{V})$. But since g is in the closure of $S(C,V)$, every neighbourhood of g intersects $S(C,V)$. In particular, there must be some $h \in S(C,V) \cap S(\{x\}, Y \setminus \overline{V})$. But this implies $h(x) \in Y \setminus \overline{V}$ and $h(x) \in h(C) \subseteq V$; a contradiction. We conclude that no such x exists, and $g \in S(C,U)$ as claimed.
We can now prove the result. Let $f \in \mathscr{C}(X,Y)$, and let $S(C,U)$ be a subbasis element containing f. Choose $c \in C$, so $f(c) \in U$. Because Y is regular, there is some open neighbourhood V_c containing $f(c)$ with $\overline{V_c} \subseteq U$. Since f is a continuous map, the preimage $f^{-1}(V_c)$ is open for each c. Since these maps cover C, and C is compact, there is finite subset $c_1, \ldots, c_k \in C$ so that $f^{-1}(V_{c_1}), \ldots, f^{-1}(V_{c_k})$ cover C.
Let $V = V_{c_1} \cup \ldots \cup V_{c_k}$. By construction,

$$f(C) \subseteq f\Big(f^{-1}(V_{c_1}) \cup \cdots \cup f^{-1}(V_{c_k})\Big) = f\Big(f^{-1}(V_{c_1} \cup \cdots \cup V_{c_k})\Big) \subseteq (V_{c_1} \cup \cdots \cup V_{c_k}) = V$$

This shows that $f \in S(C,V)$.
Moreover, $\overline{V_{c_1}} \cup \ldots \cup \overline{V_{c_k}}$ is a closed set containing V, hence

$$\overline{V} \subseteq \overline{V_{c_1}} \cup \ldots \cup \overline{V_{c_k}} \subseteq U.$$

Then, by the result above, $\overline{S(C,V)} \subseteq S(C,U)$. Thus $S(C,V)$ is the deisred sub-neighbourhood of f, which completes the proof.

31. For topological spaces X and Y, let $\mathscr{C}(X,Y)$ denote the space of continuous functions from X to Y in the compact-open topology. Suppose that Y is locally compact and Hausdorff. Show that the map

$$\begin{aligned} \mathscr{C}(X,Y) \times \mathscr{C}(Y,Z) &\to \mathscr{C}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

is continuous.

32. Prove the following.
    (a) A space X is compactly generated if and only if each set $B \subseteq X$ is closed in X if $B \cap C$ is closed for all compact C.
    (b) If X is a locally compact space, then X is compactly generated.
    (c) If X is a first countable space, then X is compactly generated.

Hint: See Munkres Lemma 46.2.

33. Let X be a compactly generated space and Y a metric space. Show that $\mathscr{C}(X, Y)$ is closed in Y^X in the topology of compact convergence.

    Hint: See Munkres Theorem 46.5.

34. Let X be a space and Y a metric space.
    (a) Show that the uniform topology on Y^X is finer than the topology of compact convergence, which is finer than the topology of pointwise convergence.
    (b) Show that, if X is compact, the topology of compact convergence and the uniform topology coincide.
    (c) Show that, if X is discrete, then the topology of compact convergence and the topology of pointwise convergence coincide.

35. Show that the set of bounded functions $\mathscr{B}(\mathbb{R}, \mathbb{R})$ is not closed in topology of compact convergence on $\mathbb{R}^{\mathbb{R}}$.

36. Let X be a locally compact Hausdorff space, and let Y be a space. We say that continuous maps $f, g : X \to Y$ are *homotopic* if there is a continuous map

$$h : X \times [0, 1] \to Y$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$. Explain the sense in which a homotopy corresponds to a continuous map

$$H : [0, 1] \to \mathscr{C}(X, Y)$$

where $\mathscr{C}(X, Y)$ has the compact-open topology.